Multi-norms and the injectivity of $L^p(G)$

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Abstract

Let G be a locally compact group, and take $p \in (1, \infty)$. We prove that the Banach left $L^1(G)$-module $L^p(G)$ is injective (if and) only if the group G is amenable. Our proof uses the notion of multi-norms. We also develop the theory of multi-normed spaces.

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1 Introduction

Let G be a locally compact group, and let $L^1(G)$ be the group algebra of G. In [5], H. G. Dales and M. E. Polyakov investigated when various canonical modules over $L^1(G)$ have certain well-known homological properties. For example, it was proved in [5, Theorem 4.9] that $L^1(G)$ is injective in $L^1(G)$-**mod**, the category of Banach left $L^1(G)$-modules, if and only if G is discrete and amenable, and in [5, Theorem 2.4] that $L^\infty(G)$ is injective in $L^1(G)$-**mod** for every locally compact group G.

One of the more difficult questions that they considered seems to have been to characterize the locally compact groups G such that the Banach left $L^1(G)$-module $L^p(G)$ is injective (for $1 < p < \infty$). By Johnson's famous theorem [14], the Banach algebra $L^1(G)$ is amenable if and only if G is an amenable group. Since $L^p(G)$ is a dual Banach $L^1(G)$-module, it follows from [11, VII.2.2] and from [22, §5.3] that $L^p(G)$ is an injective Banach left $L^1(G)$-module whenever G is amenable as a locally compact group; the converse has been an open problem for a long time. In [5], the authors obtained a partial converse to this theorem in the case where G is discrete. Indeed, they showed the following [5, Theorem 5.12]. Let G be a group, and suppose that $\ell^p(G)$ is an injective Banach left $\ell^1(G)$-module for some $p \in (1, \infty)$. Then G must be 'pseudo-amenable', a property very close to amenability. (In fact, no example of a group that is pseudo-amenable, but not amenable, is known.)

In this paper, we shall define another generalized notion of amenability, called *left* (p, q)*-amenability* of G, for any p, q such that $1 \le p \le q < \infty$ and for any locally compact group G. We shall show the following for each p, q with $1 < p \le q < \infty$:

$$L^p(G) \text{ is injective } \iff G \text{ is left } (p, q)\text{-amenable } \iff G \text{ is amenable}.$$

In particular, we resolve positively the above open problem. As a consequence, we shall also determine when the module $L^p(G)$ is flat.

In the final section §10, we shall give some similar results for the modules $\ell^p(S)$, regarded as Banach left $\ell^1(S)$-modules, for a cancellative semigroup S.

Our definition of left (p, q)-amenability is framed in the language of 'multi-norms'. The theory of multi-norms was developed by Dales and Polyakov in an attempt to resolve the above-mentioned problem. However, this theory has developed a life of its own; it is expounded at some length in [6], where many examples are given, and the connection with various known 'summing norms' and 'summing constants' is explained. We shall give a presentation of multi-norms and their duals in terms of certain tensor norms in §3 and §4.

After this paper was submitted for publication, we received the preprint [20] from Professor Gerhard Racher (Salzburg). This preprint states the following more general version of Theorem 9.6. 'Let G be a locally compact group. Suppose that there exists a non-zero, injective Banach left $L^1(G)$-module that is reflexive as a Banach space. Then G is amenable as a locally compact group.' We thank Professor Racher for sending us this preprint.

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2 Background and notation

In this section, we shall recall various notations that we shall use, and give the definitions and some properties of multi-norms and multi-bounded operators. We shall also recall the definitions of the group algebra $L^1(G)$ and the Banach left $L^1(G)$-modules $L^p(G)$ for a locally compact group G and $p \geq 1$.

2.1 Banach spaces

For $n \in \mathbb{N} = \{1, 2, \ldots\}$, we set $\mathbb{N}_n = \{1, \ldots, n\}$. The cardinality of a set S is $|S|$, and the characteristic function of a subset T of S is denoted by χ_T; we set $\delta_s = \chi_{\{s\}}$ $(s \in S)$. The conjugate to a number $p \geq 1$ is sometimes denoted by p', so that $1/p + 1/p' = 1$.

Let E be a linear space. The identity operator on E is I_E. For each $k \in \mathbb{N}$, we denote by E^k the linear space direct product of k copies of E. Let F be another linear space, and let $T : E \to F$ be a linear mapping. Then we define the linear map $T^{(k)} : E^k \to F^k$, the k^{th}-*amplification* of T, by

$$T^{(k)}(x_1, \ldots, x_k) = (Tx_1, \ldots, Tx_k) \quad (x_1, \ldots, x_k \in E)\,.$$

Let E be a normed space. Then the closed unit ball of E is denoted by $E_{[1]}$. We denote the dual space of E by E'; the action of $\lambda \in E'$ on an element $x \in E$ is written as $\langle x, \lambda \rangle$.

Let E and F be normed spaces. Then $\mathcal{B}(E, F)$ is the normed space of all bounded linear operators from E to F with the operator norm; the dual of an operator $T \in \mathcal{B}(E, F)$ is denoted by $T' \in \mathcal{B}(F', E')$. The subspaces of $\mathcal{B}(E, F)$ consisting of the finite-rank and of the compact operators are denoted by $\mathcal{F}(E, F)$ and $\mathcal{K}(E, F)$, respectively; we write $\mathcal{F}(E)$ and $\mathcal{K}(E)$ for $\mathcal{F}(E, E)$ and $\mathcal{K}(E, E)$, respectively. For $\lambda \in E'$ and $y \in F$, we define the rank-one operator $\lambda \otimes y \in \mathcal{B}(E, F)$ by

$$(\lambda \otimes y)(x) = \langle x, \lambda \rangle\, y \quad (x \in E)\,; \tag{1}$$

in this way, we identify the tensor product $E' \otimes F$ with $\mathcal{F}(E, F)$.

Let E be a normed space, and take $n \in \mathbb{N}$. Following the notation of [6] and [13], we define the *weak p-summing norm* (for $1 \leq p < \infty$) on E^n by

$$\mu_{p,n}(\boldsymbol{x}) = \sup\left\{ \left(\sum_{i=1}^{n} |\langle x_i, \lambda \rangle|^p \right)^{1/p} : \lambda \in E'_{[1]} \right\},$$

where $\boldsymbol{x} = (x_1, \ldots, x_n) \in E^n$. See also [8, p. 32] and [23, p. 134]. Notice that, by the weak*-density of $E_{[1]}$ in $E''_{[1]}$, the weak p-summing norm on $(E')^n$ can also be computed as

$$\mu_{p,n}(\boldsymbol{\lambda}) = \sup\left\{ \left(\sum_{i=1}^{n} |\langle x, \lambda_i \rangle|^p \right)^{1/p} : x \in E_{[1]} \right\}, \tag{2}$$

where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in (E')^n$.

Let K be a non-empty, locally compact space; our convention is that locally compact spaces are Hausdorff. Then $C_0(K)$ is the Banach space of complex-valued, continuous functions which vanish at infinity on K, equipped with the uniform norm $|\cdot|_K$, given by

$$|f|_K = \sup\{|f(x)| : x \in K\} \quad (f \in C_0(K)).$$

Further $C_{00}(K)$ is the subspace of $C_0(K)$ of functions with compact support.

Let (Ω, μ) be a measure space, and take $p \geq 1$. Then $L^p(\Omega) = L^p(\Omega, \mu)$ is the Banach space of (equivalence classes of) complex-valued, p-integrable functions on Ω, equipped with the norm $\|\cdot\|_p$, given by

$$\|f\|_p = \left(\int_\Omega |f|^p \,\mathrm{d}\mu\right)^{1/p} \quad (f \in L^p(\Omega)).$$

Of course, the dual of $L^p(\Omega)$ is identified with $L^{p'}(\Omega)$ when $p > 1$.

Let c_0 and ℓ^p be the usual Banach spaces. We write $(\delta_n)_{n=1}^\infty$ for the standard basis for c_0 and ℓ^p. For $n \in \mathbb{N}$, we write ℓ_n^∞ for $\mathbb{C}^n$ with the supremum norm, and we regard each ℓ_n^∞ as a subspace of c_0, and hence regard $(\delta_i)_{i=1}^n$ as a basis for ℓ_n^∞.

2.2 Banach homology

For the homological background to our work, we refer the reader to the standard reference [11]; for a clear account of all that we require, see [22, Chapter 5]. We briefly sketch what we shall need.

Let A be a Banach algebra, and let E be a Banach space that is a left A-module for the map

$$(a, x) \mapsto a \cdot x, \quad A \times E \to E.$$

Then E is a *Banach left A-module* if there is a constant $C > 0$ such that

$$\|a \cdot x\| \leq C\|a\|\|x\| \quad (a \in A,\ x \in E);$$

we denote by A-**mod** the category of Banach left A-modules. Similarly, **mod**-A and A-**mod**-A are the categories of Banach right A-modules and Banach A-bimodules, respectively. (See [2], [11], [14], and [22], for example.)

Let $E \in A$-**mod**. Then E is *essential* if the linear span of the elements $a \cdot x$ for $a \in A$ and $x \in E$ is dense in E. In the case where A has a bounded left approximate identity, this implies [2, Corollary 2.9.26] that E is *neo-unital*, in the sense that each element in E has the form $a \cdot x$ for some $a \in A$ and $x \in E$.

Let $E \in A$-**mod**. Then the dual action of A on E' is defined by

$$\langle x, \lambda \cdot a \rangle = \langle a \cdot x, \lambda \rangle \quad (a \in A,\ x \in E,\ \lambda \in E'),$$

and then $E' \in$ **mod**-A is the *dual module to* E. Similarly, $E' \in A$-**mod** when $E \in$ **mod**-A.

For spaces $E, F \in A$-**mod**, the Banach space of bounded A-module morphisms from E to F is denoted by ${}_A\mathcal{B}(E, F)$. A monomorphism $T \in {}_A\mathcal{B}(E, F)$ is said to be *admissible* if there exists $S \in \mathcal{B}(F, E)$ such that $S \circ T = I_E$, and T is a *coretraction* if there exists $S \in {}_A\mathcal{B}(F, E)$ such that $S \circ T = I_E$.

Definition 2.1. Let A be a Banach algebra, and let $J \in A$-**mod**. Then J is *injective* if, for each $E, F \in A$-**mod**, for each admissible monomorphism $T \in {}_A\mathcal{B}(E, F)$, and for each $S \in {}_A\mathcal{B}(E, J)$, there exists $R \in {}_A\mathcal{B}(F, J)$ such that $R \circ T = S$.

Let A be a Banach algebra, and let E be a Banach space. Then $\mathcal{B}(A,E) \in A$-**mod** when we define the module operation by the formula

$$(a \cdot T)(b) = T(ba) \quad (a, b \in A,\, T \in \mathcal{B}(A,E)).$$

Now suppose that $E \in A$-**mod**. Then we define the *canonical embedding* $\Pi : E \to \mathcal{B}(A,E)$ by the formula

$$\Pi(x)(a) = a \cdot x \quad (a \in A,\, x \in E),$$

so that $\Pi \in {}_A\mathcal{B}(E, \mathcal{B}(A,E))$. The mapping Π is indeed an embedding if E has the property that $x = 0$ whenever $x \in E$ and $a \cdot x = 0$ for all $a \in A$; i.e., $\{x \in E : A \cdot x = \{0\}\} = \{0\}$. This property holds whenever A has a bounded left approximate identity and E is essential.

For background, we note the following characterization of injective modules [5, Proposition 1.7]; we shall use related ideas.

Proposition 2.2. *Let A be a Banach algebra, and let $E \in A$-**mod** have the property that $\{x \in E : A \cdot x = \{0\}\} = \{0\}$. Then the module E is injective if and only if the morphism $\Pi \in {}_A\mathcal{B}(E, \mathcal{B}(A,E))$ is a coretraction in A-**mod**.* □

We shall take the following as our definition of a flat module; a different, more intrinsic, definition is given in [11, VI.1.1] and [22, Definition 5.3.3], and the equivalence of the two definitions is shown in [11, VII.1.14] and [22, Theorem 5.3.8].

Definition 2.3. Let A be a Banach algebra, and let $E \in A$-**mod**. Then E is *flat* if the dual module E' is injective in **mod**-A.

We note that every projective module in A-**mod** is flat [22, Examples 5.3.9(b)]. We shall use the following basic result of Helemskii [10]: see [11, VII.2.29] and [22, Theorem 5.3.8 and Example 5.3.9(a)]. The notion of an amenable Banach algebra originates with Johnson [14]; see [2, §2.8], [11], and [22].

Theorem 2.4. *Let A be an amenable Banach algebra. Then every dual module in A-**mod** or **mod**-A is injective; equivalently, every module in A-**mod** or **mod**-A is flat.* □

2.3 L^p modules over group algebras

Let G be a locally compact group with left Haar measure m and modular function Δ, and set $L^1(G) = L^1(G,m)$; see [2, §3.3]. For $f \in L^1(G)$ and $s \in G$, we define $s \cdot f \in L^1(G)$ by

$$(s \cdot f)(t) = f(s^{-1}t) \quad (t \in G),$$

so defining an action of G on the space $L^1(G)$. We can extend this action by duality to the space $L^\infty(G)' = L^1(G)''$. An element $\Lambda \in L^\infty(G)'$ is a *mean* on $L^\infty(G)$ if

$$\langle 1, \Lambda \rangle = \|\Lambda\| = 1,$$

and Λ is *left-invariant* if $\{s \cdot \Lambda : s \in G\} = \{\Lambda\}$. The group G is *amenable* if there exists a left-invariant mean on $L^\infty(G)$. There are many different characterizations of the amenability of G; see [17], for example, for a full account.

Let G be a locally compact group. We now consider $L^1(G)$ as a Banach algebra equipped with the *convolution product* $\star$ given by

$$(f \star g)(s) = \int_G f(t)g(t^{-1}s)\,\mathrm{d}m(t) \quad (s \in G), \tag{3}$$

where $f, g \in L^1(G)$ and the integral is defined for almost all $s \in G$. It is standard that $(L^1(G), \star)$ has a bounded approximate identity. It is a very famous theorem of Johnson [14] that the algebra $L^1(G)$ is

amenable as a Banach algebra if and only if the locally compact group G is amenable; see also [2, Theorem 5.6.42].

We denote by φ_G the *augmentation character* on G, given by

$$\varphi_G(f) = \int_G f(t)\,\mathrm{d}m(t) \quad (f \in L^1(G))\,.$$

Let $p \in [1,\infty)$, and set $E = L^p(G) = L^p(G,m)$. Take $f \in L^1(G)$ and $g \in L^p(G)$. Then again we can define $f \star g$ on G via (3), and in this case we have $f \star g \in L^p(G)$. With this multiplication, $L^p(G)$ has the structure of a Banach left $L^1(G)$-module; indeed, we have $L^p(G) \in L^1(G)$-**mod** [2, Theorem 3.3.19]. The module E is essential, and so Proposition 2.2 applies.

In fact, the spaces $L^p(G)$ are Banach $L^1(G)$-bimodules, where the right module action of $L^1(G)$ on $L^p(G)$ is defined as

$$(g \star f)(s) = \int g(st^{-1})f(t)\Delta^{1/p}(t^{-1})\,\mathrm{d}m(t) \quad (s \in G)$$

for $f \in L^1(G)$ and $g \in L^p(G)$. This formula in the case where $p = 1$ gives the same right action as the convolution product on $L^1(G)$.

We shall use the notation $\cdot$ for the module products on $L^{p'}(G)$ considered as the dual module of $(L^p(G), \star)$; the formulae for these products are given in [2, §3.3]. These dual module actions are similar to, but different from, the standard actions $\star$ defined above.

2.4 Multi-normed spaces

The following definition is due to Dales and Polyakov. For a full account of the theory of multi-normed spaces, see [6].

Definition 2.5. Let $(E, \|\cdot\|)$ be a normed space, and let $(\|\cdot\|_n : n \in \mathbb{N})$ be a sequence such that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$, with $\|\cdot\|_1 = \|\cdot\|$ on E. Then the sequence $(\|\cdot\|_n : n \in \mathbb{N})$ is a *multi-norm* if the following axioms hold (where in each case the axiom is required to hold for all $n \geq 2$ and all $x_1, \dots, x_n \in E$):

(A1) $\|(x_{\sigma(1)}, \dots, x_{\sigma(n)})\|_n = \|(x_1, \dots, x_n)\|_n$ for each permutation σ of $\mathbb{N}_n$;

(A2) $\|(\alpha_1 x_1, \dots, \alpha_n x_n)\|_n \leq \max_{i\in\mathbb{N}_n} |\alpha_i|\, \|(x_1, \dots, x_n)\|_n \quad (\alpha_1, \dots, \alpha_n \in \mathbb{C})\,$;

(A3) $\|(x_1, \dots, x_{n-1}, 0)\|_n = \|(x_1, \dots, x_{n-1})\|_{n-1}\,$;

(A4) $\|(x_1, \dots, x_{n-2}, x_{n-1}, x_{n-1})\|_n = \|(x_1, \dots, x_{n-2}, x_{n-1})\|_{n-1}\,$.

The normed space E equipped with a multi-norm is a *multi-normed space*, denoted in full by $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$. We say that such a multi-norm is *based on* E.

Suppose that in the above definition we replace axiom (A4) by the following axiom:

(B4) $\|(x_1, \dots, x_{n-2}, x_{n-1}, x_{n-1})\|_n = \|(x_1, \dots, x_{n-2}, 2x_{n-1})\|_{n-1}$.

Then we obtain the definition of a *dual multi-norm* and of a *dual multi-normed space*. (A yet more general concept, that of sequences $(\|\cdot\|_n : n \in \mathbb{N})$ satisfying just (A1)–(A3), is mentioned in [6, §2.2.1].)

Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed or dual multi-normed space. For each $n \in \mathbb{N}$, the dual of the space $(E^n, \|\cdot\|_n)$ can be isomorphically identified with the Banach space $(E')^n$, as explained in [6, §1.2.4], and in this way we regard $(E')^n$ as a Banach space. The weak* topology from this duality is the product of the weak* topologies given by the duality of E and E'.

The following results are noted in [6, Chapter 2]. First, the axioms (A1)–(A4) are independent [6, §2.1.3]. Second, in the case where $(\|\cdot\|_n : n \in \mathbb{N})$ satisfies (A1)–(A3), we have

$$\max_{i\in\mathbb{N}_n} \|x_i\| \leq \|(x_1, \dots, x_n)\|_n \leq \sum_{i=1}^n \|x_i\| \quad (x_1, \dots, x_n \in E)$$

for each $n \in \mathbb{N}$, and so $\|\cdot\|_n$ defines the same topology on E^n as the product topology [6, Lemma 2.11]. Third, if $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-norm or a dual multi-norm based on E, and $\|\cdot\|_n'$ is the dual norm to $\|\cdot\|_n$ for each $n \in \mathbb{N}$, then $(\|\cdot\|_n' : n \in \mathbb{N})$ is a dual multi-norm or multi-norm, respectively, based on E' [6, §2.3.2]. This latter result implies that the sequence of second duals of a multi-norm $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-norm based on E''.

The family $\mathcal{E}_E$ of all multi-norms based on a normed space E is a Dedekind complete lattice with respect to the ordering $\leq$, where $(\|\cdot\|_n^1 : n \in \mathbb{N}) \leq (\|\cdot\|_n^2 : n \in \mathbb{N})$ if

$$\|\boldsymbol{x}\|_n^1 \leq \|\boldsymbol{x}\|_n^2 \quad (\boldsymbol{x} \in E^n,\ n \in \mathbb{N})$$

[6, Proposition 3.10]. The minimum element of the lattice $(\mathcal{E}_E, \leq)$ is the *minimum multi-norm* $(\|\cdot\|_n^{\min} : n \in \mathbb{N})$, and the formula for $\|\cdot\|_n^{\min}$ is

$$\|(x_1, \dots, x_n)\|_n^{\min} = \max_{i \in \mathbb{N}_n} \|x_i\| \quad (x_1, \dots, x_n \in E)$$

for each $n \in \mathbb{N}$, as in [6, Definition 3.2].

For each normed space E, there is a unique maximum element in the lattice $(\mathcal{E}_E, \leq)$; this is the *maximum multi-norm* $(\|\cdot\|_n^{\max} : n \in \mathbb{N})$. By [6, Theorem 3.33], for each $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$ and each $n \in \mathbb{N}$, we have

$$\|\boldsymbol{x}\|_n^{\max} = \sup \left\{ \left| \sum_{i=1}^n \langle x_i, \lambda_i \rangle \right| : \lambda_1, \dots, \lambda_n \in E',\ \mu_{1,n}(\lambda_1, \dots, \lambda_n) \leq 1 \right\}. \tag{4}$$

Further, $\mu_{1,n}$ on $(E')^n$ is the dual norm to the norm $\|\cdot\|_n^{\max}$ on E^n.

A multi-norm $(\|\cdot\|_n^2 : n \in \mathbb{N})$ in $\mathcal{E}_E$ *dominates* a multi-norm $(\|\cdot\|_n^1 : n \in \mathbb{N})$ in $\mathcal{E}_E$ if there is a constant $C > 0$ such that $\|\boldsymbol{x}\|_n^1 \leq C\|\boldsymbol{x}\|_n^2 \quad (\boldsymbol{x} \in E^n,\ n \in \mathbb{N})$; two multi-norms are *equivalent* if each dominates the other.

The following is [6, Definition 6.4] (where c_B is used instead of our $\mathrm{mb}(B)$ to denote the multi-bound of a set B).

Definition 2.6. Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space. A subset $B \subset E$ is *multi-bounded* if

$$\mathrm{mb}(B) := \sup \{\|(x_1, \dots, x_n)\|_n : x_1, \dots, x_n \in B,\ n \in \mathbb{N}\} < \infty .$$

The constant $\mathrm{mb}(B)$ is the *multi-bound* of B.

The following easy remark is [6, Proposition 6.5(ii)].

Lemma 2.7. *Let E be a multi-normed space. Then the absolutely convex hull of a multi-bounded set is multi-bounded, with the same multi-bound.* □

Definition 2.8. Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ and $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be multi-normed spaces, and let $T \in \mathcal{B}(E, F)$. Then T is *multi-bounded* if

$$\|T\|_{mb} := \sup_{k \in \mathbb{N}} \|T^{(k)}\| < \infty .$$

We set

$$\mathcal{M}(E, F) = \{T \in \mathcal{B}(E, F) : \|T\|_{mb} < \infty\} ,$$

so that $\mathcal{M}(E, F)$ is the space of *multi-bounded operators*.

Here, $\|T^{(k)}\|$ is calculated by regarding $T^{(k)}$ as a bounded linear map from $(E^k, \|\cdot\|_k)$ into $(F^k, \|\cdot\|_k)$. It is easy to check that $(\mathcal{M}(E,F), \|\cdot\|_{mb})$ is a normed space, and that it is a Banach space in the case where F is a Banach space; $\|\cdot\|_{mb}$ is the *multi-bounded norm* on $\mathcal{M}(E,F)$.

Let E and F be multi-normed spaces, and let $T \in \mathcal{M}(E,F)$. It follows immediately from the definitions that $T(B)$ is a multi-bounded set in F whenever B is a multi-bounded set in E. Conversely, it is noted in [6, §6.1.3] that any $T \in \mathcal{B}(E,F)$ which takes multi-bounded sets to multi-bounded sets is multi-bounded, and, further, that

$$\|T\|_{mb} = \sup\{\mathrm{mb}[T(B)] : B \subset E,\ \mathrm{mb}(B) \le 1\}\ .$$

Thus our definitions of $\mathcal{M}(E,F)$ and $\|\cdot\|_{mb}$ are equivalent to those given in [6, Definitions 6.9 and 6.12]; the definitions in [6] apply more generally.

Proposition 2.9. *Let E be a multi-normed space, and consider ℓ^1 with its minimum multi-norm. Then, for each $T \in \mathcal{B}(\ell^1, E)$, we have*

$$\|T\|_{mb} = \mathrm{mb}\{T(\delta_k) :\ k \in \mathbb{N}\} = \mathrm{mb}\ T(\ell^1_{[1]})\,,$$

so that $\mathcal{M}(\ell^1, E) = \left\{T \in \mathcal{B}(\ell^1, E) :\ \mathrm{mb}\{T(\delta_k) :\ k \in \mathbb{N}\} < \infty\right\}$.

Proof. This follows directly from Lemma 2.7 and the previous paragraph. □

The following result is immediate; a more general result is given in [6, Theorem 6.17].

Proposition 2.10. *Let $((E^n, \|\cdot\|_n) :\ n \in \mathbb{N})$ and $((F^n, \|\cdot\|_n^{\min}) :\ n \in \mathbb{N})$ be multi-normed spaces. Then each $T \in \mathcal{B}(E,F)$ is multi-bounded and $\|T\|_{mb} = \|T\|$.* □

For normed spaces E and F, the normed space of nuclear operators from E to F is denoted by $(\mathcal{N}(E,F), \nu)$, where ν is the nuclear norm. It is shown in [6, Theorem 6.15(ii)] that there is a natural contractive inclusion

$$(\mathcal{N}(E,F), \nu) \hookrightarrow (\mathcal{M}(E,F), \|\cdot\|_{mb})\,.$$

In particular, $\mathcal{F}(E,F) \subset \mathcal{M}(E,F)$. It is shown in [6, Example 6.25] that the 'minimum case', where $\mathcal{M}(E,F) = \mathcal{N}(E,F)$, can occur.

Example 2.11. There are many examples of multi-normed spaces in [6]; we shall give some below. An important example is the lattice multi-norm described in [6, §4.3]. Indeed, let $(E, \|\cdot\|)$ be a (complex) Banach lattice. For $n \in \mathbb{N}$, set

$$\|(x_1, \dots, x_n)\|_n^L = \|\,|x_1| \vee \cdots \vee |x_n|\,\| \qquad (x_1, \dots, x_n \in E)\,.$$

Then it is easily checked that $(\|\cdot\|_n^L :\ n \in \mathbb{N})$ is a multi-norm based on E; it is the *lattice multi-norm.* □

3 Multi-normed spaces as tensor norms

In this section, we shall show that there are bijections between the families of multi-norm structures based on a normed space E and certain families of norms on the tensor products $c_0 \otimes E$ and $\ell^\infty \otimes E$.

Suppose that E and F are normed spaces, and that $\|\cdot\|$ is a norm on $E \otimes F$. Then $\|\cdot\|$ is a *sub-cross-norm* if $\|x \otimes y\| \le \|x\|\|y\|$ $(x \in E,\ y \in F)$, and a *cross-norm* if

$$\|x \otimes y\| = \|x\|\|y\| \quad (x \in E,\ y \in F)\,.$$

Further, a sub-cross-norm on $E \otimes F$ is *reasonable* if the linear functional $\lambda \otimes \mu$ is bounded, with $\|\lambda \otimes \mu\| \le \|\lambda\|\|\mu\|$, for each $\lambda \in E'$ and $\mu \in F'$. In fact, each reasonable sub-cross-norm on $E \otimes F$ is a cross-norm, and $\|\lambda \otimes \mu\| = \|\lambda\|\|\mu\|$ for each $\lambda \in E'$ and $\mu \in F'$.

The *injective tensor norm* $\|\cdot\|_\varepsilon$ on $E \otimes F$ is defined by identifying $E \otimes F$ with a subspace of $\mathcal{B}(E', F)$; here $x \otimes y$ corresponds to the map $\lambda \mapsto \langle x, \lambda \rangle\, y,\; E' \to F$. The completion of $E \otimes F$ with respect to this norm is the *injective tensor product*, denoted by $E \check{\otimes} F$. The *projective tensor norm* $\|\cdot\|_\pi$ on $E \otimes F$ is defined by

$$\|\tau\|_\pi = \inf \sum_{j=1}^{n} \|x_j\| \|y_j\| \,,$$

where the infimum is taken over all representations $\tau = \sum_{j=1}^{n} x_j \otimes y_j$ of τ in $E \otimes F$; the completion of $E \otimes F$ with respect to this norm is the *projective tensor product*, denoted by $E \widehat{\otimes} F$. The norms $\|\cdot\|_\varepsilon$ and $\|\cdot\|_\pi$ are both reasonable cross-norms on $E \otimes F$, and a norm $\|\cdot\|$ on $E \otimes F$ is a reasonable cross-norm if and only if

$$\|z\|_\varepsilon \leq \|z\| \leq \|z\|_\pi \quad (z \in E \otimes F)\,.$$

For $\mu \in (E \widehat{\otimes} F)'$, define $T_\mu \in \mathcal{B}(E, F')$ by

$$\langle y, T_\mu x \rangle = \langle x \otimes y, \mu \rangle \quad (x \in E,\ y \in F)\,.$$

Then the map $\mu \mapsto T_\mu,\; (E \widehat{\otimes} F)' \to \mathcal{B}(E, F')$, is an isometric isomorphism; we shall identify $(E \widehat{\otimes} F)'$ and $\mathcal{B}(E, F')$. See [7, Chapter II] and [23, §6.1] for accounts of tensor norms on $E \otimes F$ that include the above remarks.

The following characterization of multi-norms is given in [6, Theorem 2.35]. (There is a similar characterization of dual multi-norms in [6, Theorem 2.36].)

Theorem 3.1. *Let $(E, \|\cdot\|)$ be a normed space, and suppose that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$, with $\|x\|_1 = \|x\|$ $(x \in E)$. Then the following are equivalent:*

(a) $(\|\cdot\|_n : n \in \mathbb{N})$ *is a multi-norm on E;*

(b) $\|Tx\|_m \leq \|T\| \|x\|_n$ *for each $T \in \mathcal{B}(\ell^\infty_n, \ell^\infty_m)$, each $x \in E^n$, and each $m, n \in \mathbb{N}$.* □

We shall now characterize multi-norm spaces in terms of single norms on a certain tensor product; an analogous 'coordinate-free' characterization of operator spaces is developed in [12] and [18].

Definition 3.2. Let E be a normed space. Then a norm $\|\cdot\|$ on $c_0 \otimes E$ is a *c_0-norm* if $\|\delta_1 \otimes x\| = \|x\|$ for each $x \in E$ and if the linear operator $T \otimes I_E$ is bounded on $(c_0 \otimes E, \|\cdot\|)$ with norm at most $\|T\|$ for each $T \in \mathcal{K}(c_0)$.

Similarly, a norm $\|\cdot\|$ on $\ell^\infty \otimes E$ is an *ℓ^∞-norm* if $\|\delta_1 \otimes x\| = \|x\|$ for each $x \in E$ and if the linear operator $T \otimes I_E$ is bounded on $(\ell^\infty \otimes E, \|\cdot\|)$ with norm at most $\|T\|$ for each $T \in \mathcal{K}(\ell^\infty)$.

Note that, in the definition of c_0-norms or ℓ^∞-norms above, we use $\mathcal{K}(c_0)$ and $\mathcal{K}(\ell^\infty)$, respectively. However, we shall soon see that we can replace $\mathcal{K}(c_0)$ and $\mathcal{K}(\ell^\infty)$ by the larger spaces $\mathcal{B}(c_0)$ and $\mathcal{B}(\ell^\infty)$, respectively.

Lemma 3.3. *Let E be a normed space. Then each c_0-norm on $c_0 \otimes E$ and each ℓ^∞-norm on $\ell^\infty \otimes E$ is a reasonable cross-norm.*

Proof. Suppose that $\|\cdot\|$ is a c_0-norm on $c_0 \otimes E$. First, given $a, b \in c_0$ of the same norm and given $x \in E$, we see that $\|b \otimes x\| \leq \|a \otimes x\|$ by considering the rank-one operator of norm 1 in $\mathcal{K}(c_0)$ which maps a to b. It follows that

$$\|a \otimes x\| = \|a\| \|\delta_1 \otimes x\| = \|a\| \|x\| \quad (a \in c_0,\ x \in E)\,.$$

From this, it follows from the triangle inequality that $\|\tau\| \leq \|\tau\|_\pi$ $(\tau \in c_0 \otimes E)$.

Let $\tau = \sum_{j=1}^{k} b_j \otimes x_j \in c_0 \otimes E \subset \ell^\infty \otimes E$; as in equation (1), τ is identified with the map

$$\tau : f \mapsto \sum_{j=1}^{k} \langle f, b_j \rangle \, x_j \quad \text{in} \quad \mathcal{F}(\ell^1, E) \subset \mathcal{B}(\ell^1, E)\,.$$

Take $f \in \ell^1$, and consider the rank-one operator $T \in \mathcal{K}(c_0)$ defined by setting $Tb = \langle f, b \rangle \, \delta_1$ for $b \in c_0$, where we regard δ_1 as an element of c_0. Then we see that $\|T\| = \|f\|$ and

$$\|\tau(f)\| = \left\| \sum_{j=1}^{k} \langle f, b_j \rangle \, x_j \right\| = \left\| \sum_{j=1}^{k} \langle f, b_j \rangle \, \delta_1 \otimes x_j \right\| = \|(T \otimes I_E)(\tau)\| \leq \|T\| \|\tau\| = \|f\| \|\tau\|\,,$$

using the fact that $\|\cdot\|$ is a c_0-norm on $c_0 \otimes E$. We conclude that $\|\tau\|_\varepsilon \leq \|\tau\|$.

Thus $\|\tau\|_\varepsilon \leq \|\tau\| \leq \|\tau\|_\pi$ for each $\tau \in c_0 \otimes E$, and so $\|\cdot\|$ is a reasonable cross-norm on $c_0 \otimes E$.

The case of an ℓ^∞-norm on $\ell^\infty \otimes E$ can be dealt with similarly. □

Theorem 3.4. *Let E be a normed space. Then there exist bijective correspondences between:*

1. *the collection of multi-norms based on E;*
2. *the collection of norms $\|\cdot\|$ on $\mathcal{F}(\ell^1, E)$ with the properties that $\|\delta_1 \otimes x\| = \|x\|$ for each $x \in E$ and that*

$$\|S \circ T\| \leq \|T : \ell^1 \to \ell^1\| \|S\| \quad (T \in \mathcal{B}(\ell^1),\ S \in \mathcal{F}(\ell^1, E))\,; \tag{5}$$

3. *the collection of c_0-norms on $c_0 \otimes E$; and*
4. *the collection of ℓ^∞-norms on $\ell^\infty \otimes E$.*

Proof. Let $(\|\cdot\|_n : \ n \in \mathbb{N})$ be a multi-norm based on E. Consider the minimum multi-norm on ℓ^1, so that, by Proposition 2.10, $\mathcal{M}(\ell^1) = \mathcal{B}(\ell^1)$ and $\|T\|_{mb} = \|T\|$ for $T \in \mathcal{B}(\ell^1)$. Since $\mathcal{F}(\ell^1, E) \subset \mathcal{M}(\ell^1, E)$, we can consider the norm $\|\cdot\|$ on $\mathcal{F}(\ell^1, E)$ to be the restriction of the norm $\|\cdot\|_{mb}$ of $\mathcal{M}(\ell^1, E)$. For each $x \in E$, we have, by Proposition 2.9,

$$\|\delta_1 \otimes x\| = \|\delta_1 \otimes x\|_{mb} = \mathrm{mb}\,\{(\delta_1 \otimes x)(\delta_k) : \ k \in \mathbb{N}\} = \|x\|\,.$$

It now follows easily that the norm $\|\cdot\|$ on $\mathcal{F}(\ell^1, E)$ satisfies (5), and hence $\|\cdot\|$ satisfies the conditions in clause (ii).

Since $c_0 \otimes E \subset \mathcal{F}(\ell^1, E)$, once we have a norm $\|\cdot\|$ on $\mathcal{F}(\ell^1, E)$ with the properties as stated in clause (ii), we can give $c_0 \otimes E$ the norm which is the restriction of the norm $\|\cdot\|$ on $\mathcal{F}(\ell^1, E)$. It is easily checked that this norm is a c_0-norm on $c_0 \otimes E$; in fact, for each $T \in \mathcal{B}(c_0)$, the linear operator $T \otimes I_E$ is bounded on $(c_0 \otimes E, \|\cdot\|)$ with norm at most $\|T\|$.

Using the identification of $\mathcal{F}(\ell^1, E)$ with $\ell^\infty \otimes E$ given in equation (1), we can give $\ell^\infty \otimes E$ a norm $\|\cdot\|$ with properties similar to those of the ℓ^∞-norms, except that $\mathcal{K}(\ell^\infty)$ is replaced by the set $\{S' : \ S \in \mathcal{B}(\ell^1)\}$. Let $T \in \mathcal{B}(\ell^\infty)$ and $\sigma \in \ell^\infty \otimes E$. We wish to show that

$$\|(T \otimes I_E)(\sigma)\| \leq \|T\| \|\sigma\|\,.$$

Indeed, suppose that $\sigma = \sum_{i=1}^{n} a_i \otimes x_i$, where $a_1, \dots, a_n \in \ell^\infty$ and $x_1, \dots, x_n \in E$, and take $\varepsilon > 0$. Let F be the linear span of the set $\{a_1, \dots, a_n\}$ in ℓ^∞, set $G = T(F)$, and consider $T_0 : F \to G$ to be the restriction of T. We can identify F' and G' as ℓ^1/X and ℓ^1/Y, respectively, where

$$X = \{f \in \ell^1 : \ f \,|\, F = 0\} \quad \text{and} \quad Y = \{f \in \ell^1 : \ f \,|\, G = 0\}$$

are closed subspaces of finite codimension in ℓ^1. By the projectivity of ℓ^1 [7, p. 72], there is an operator $S \in \mathcal{B}(\ell^1)$ such that the following diagram commutes:

$$\begin{tikzcd} \ell^1 \arrow[r, "S"] \arrow[d, two heads, "\pi_Y"'] & \ell^1 \arrow[d, two heads, "\pi_X"] \\ \ell^1/Y = G' \arrow[r, "T_0'"] & \ell^1/X = F' \end{tikzcd};$$

moreover, we can choose S such that $\|S\| \leq \|T_0'\| + \varepsilon$. It follows that S' and T agree on F and that $\|S\| \leq \|T\| + \varepsilon$. Thus

$$\|(T \otimes I_E)(\sigma)\| = \|(S' \otimes I_E)(\sigma)\| \leq \|S\|\|\sigma\| \leq (\|T\| + \varepsilon)\|\sigma\|.$$

Letting $\varepsilon \searrow 0$, we obtain the desired inequality. In particular, this shows that $\|\cdot\|$ is an ℓ^∞-norm on $\ell^\infty \otimes E$.

Thus, we have constructed maps from the collection of multi-norms based on E into the collections specified in the clauses (i), (ii), and (iii).

To define the (proposed) inverses of these maps, first suppose that we already have a c_0-norm $\|\cdot\|$ on $c_0 \otimes E$. Then we define

$$\|(x_1, x_2, \ldots, x_n)\|_n = \left\|\sum_{j=1}^n \delta_j \otimes x_j\right\| \qquad \big(x_1, \ldots, x_n \in E,\ n \in \mathbb{N}\big).$$

Clearly $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$, and it is easy to see that clause (b) of Theorem 3.1 is satisfied, and so it follows from Theorem 3.1 that $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-norm on E.

To prove that the correspondences defined above are bijections, it is sufficient to show that any given multi-norm $(\|\cdot\|_n :\ n \in \mathbb{N})$ based on E determines uniquely an ℓ^∞-norm $\|\cdot\|$ on $\ell^\infty \otimes E$ and a c_0-norm $\|\cdot\|$ on $c_0 \otimes E$ such that

$$\left\|\sum_{j=1}^n \delta_j \otimes x_j\right\| = \|(x_1, x_2, \ldots, x_n)\|_n \qquad \big(x_1, \ldots, x_n \in E,\ n \in \mathbb{N}\big). \tag{6}$$

So, let $\sigma = \sum_{i=1}^n a_i \otimes x_i$ be an element of $\ell^\infty \otimes E$, and take $\varepsilon > 0$. Let F be the linear span of the set $\{a_1, \ldots, a_n\}$ in ℓ^∞. Then there exist $N \in \mathbb{N}$ and a subspace G of ℓ_N^∞ such that the Banach–Mazur distance $d(F, G) < 1 + \varepsilon$, and so there exists $T_0 : F \to G$ with $\|T_0\| < 1+\varepsilon$ and $\|T_0^{-1}\| = 1$. The injectivity of ℓ_N^∞ and ℓ^∞ then implies that T_0 and T_0^{-1} extend to linear operators T and S, respectively, in $\mathcal{B}(\ell^\infty)$ with $\|T\| = \|T_0\| < 1 + \varepsilon$ and $\|S\| = \|T_0^{-1}\| = 1$ and with the range of T contained in ℓ_N^∞. For each $i \in \mathbb{N}_n$, set $a_{i,\varepsilon} = T_0 a_i$, and then set

$$\sigma_\varepsilon = \sum_{i=1}^n a_{i,\varepsilon} \otimes x_i \in \ell_N^\infty \otimes E.$$

It follows that

$$\sigma_\varepsilon = (T \otimes I_E)(\sigma) \quad \text{and} \quad \sigma = (S \otimes I_E)(\sigma_\varepsilon) = (STS \otimes I_E)(\sigma_\varepsilon).$$

Note that both T and STS belong to $\mathcal{K}(\ell^\infty)$, and so the ℓ^∞-norm property implies that

$$(1 + \varepsilon)^{-1}\|\sigma\| \leq \|\sigma_\varepsilon\| \leq (1 + \varepsilon)\|\sigma\|.$$

Thus we obtain $\|\sigma\| = \lim_{\varepsilon \searrow 0} \|\sigma_\varepsilon\|$. Further note that, since $\sigma_\varepsilon \in c_{00} \otimes E$, by (6), its norm is totally determined by the multi-norm $(\|\cdot\|_n :\ n \in \mathbb{N})$. Hence, the ℓ^∞-norm $\|\cdot\|$ on $\ell^\infty \otimes E$ is determined completely by the given multi-norm on E.

For the case of a c_0-norm $\|\cdot\|$ on $c_0 \otimes E$, notice that, for each $\sigma \in c_0 \otimes E$, we have

$$\|\sigma\| = \lim_{n\to\infty} \|(P_n \otimes I_E)(\sigma)\|,$$

where $P_n \in \mathcal{K}(c_0)$ is the projection onto the first n coordinates. We see, again by (6), that the norm of $(P_n \otimes I_E)(\sigma)$ is determined by the multi-norm $(\|\cdot\|_n : \ n \in \mathbb{N})$. ☐

Thus, in particular, the study of multi-norms based on a normed space E is equivalent to the study of c_0-norms on $c_0 \otimes E$.

Remark 3.5. Let E be a multi-normed space with the associated c_0-norm $\|\cdot\|$ on $c_0 \otimes E$. From the theorem above, it follows that, for each $\sigma \in c_0 \otimes E$, we have

$$\|\sigma\| = \|\sigma\|_{\mathcal{M}(\ell^1,E)} = \mathrm{mb}\,\{\sigma(\delta_k) : \ k \in \mathbb{N}\}\ ,$$

where, in the last two terms, σ is considered as an element of $\mathcal{F}(\ell^1, E)$.

Theorem 3.4 and its proof imply the following.

Corollary 3.6. *Let E be a normed space.*

1. *Suppose that $\|\cdot\|$ is a c_0-norm on $c_0 \otimes E$. Then, for each $T \in \mathcal{B}(c_0)$, the operator $T \otimes I_E$ is bounded on $(c_0 \otimes E, \|\cdot\|)$, with norm $\|T\|$.*
2. *Suppose that $\|\cdot\|$ is an ℓ^∞-norm on $\ell^\infty \otimes E$. Then, for each $T \in \mathcal{B}(\ell^\infty)$, the operator $T \otimes I_E$ is bounded on $(\ell^\infty \otimes E, \|\cdot\|)$, with norm $\|T\|$.* ☐

Corollary 3.7. *Let E be a normed space. The maximum multi-norm structure based on E corresponds to the projective tensor norm on $c_0 \otimes E$, and the minimum multi-norm structure based on E corresponds to the injective tensor norm on $c_0 \otimes E$.* ☐

A related result about the maximum multi-norm is proved by more elementary calculations in [6, Theorem 3.43].

Remark 3.8. As discussed in [6, §2.4.5], it follows from the previous corollary that our notion of a c_0-norm is equivalent to that of a norm on $c_0 \otimes E$ satisfying 'condition (P)' of Pisier. This condition is the main topic of the memoir [15]. The following theorem gives a general *representation theorem for multi-normed spaces*. It shows a universal property of the lattice multi-norms described in Example 2.11; the result follows from a theorem of Pisier stated as [15, Théorème 2.1]. We are indebted to the late Professor Nigel Kalton for the reference to [15].

Theorem 3.9. *Let $((E_n, \|\cdot\|_n) : \ n \in \mathbb{N})$ be a multi-Banach space. Then there is a Banach lattice X and an isometric embedding $J : E \to X$ such that, for each $n \in \mathbb{N}$, we have*

$$\|(Jx_1, \dots, Jx_n)\|_n^L = \|(x_1, \dots, x_n)\|_n \qquad (x_1, \dots, x_n \in E)\,.$$ ☐

Proposition 3.10. *Let E be a multi-normed space, and assume that $\mathcal{M}(\ell^1, E)$ consists of weakly compact operators. Then every multi-bounded subset of E is relatively weakly compact.*

Proof. By the Eberlein-Šmulian theorem [16, 2.4.6], we need to consider only a countable multi-bounded subset $\{x_n : n \in \mathbb{N}\}$ of E. In this case, it then follows that the map $T : \delta_n \mapsto x_n$ extends to a bounded linear operator $T : \ell^1 \to E$ which in fact belongs to $\mathcal{M}(\ell^1, E)$. Thus, by the assumption, T must be weakly compact. In particular, $\{x_n : n \in \mathbb{N}\} \subset T(\ell^1_{[1]})$ is relatively weakly compact. ☐

4 Dual multi-normed spaces

In this section, we shall quickly sketch how *dual multi-normed spaces* (see [6, §2.1.2]) fit into a tensor–norm framework.

Definition 4.1. Let E be a normed space. Then a norm $\|\cdot\|$ on $\ell^1 \otimes E$ is an *ℓ^1-norm* if $\|\delta_1 \otimes x\| = \|x\|$ for each $x \in E$ and if the linear operator $T \otimes I_E$ is bounded on $(\ell^1 \otimes E, \|\cdot\|)$ with norm at most $\|T\|$ for each $T \in \mathcal{K}(\ell^1)$.

Note that, again, an ℓ^1-norm on $\ell^1 \otimes E$ is necessarily a reasonable cross-norm.

In fact, there is an analogue of Theorem 3.4 that relates dual multi-norms based on a normed space E to ℓ^1-norms on $\ell^1 \otimes E$. We shall not use this result, and so omit the details, but we note that the necessary preliminary results are contained in [6, §2.3.2], where the basic results on the relations between multi-norms and dual multi-norms are obtained in a different way.

Let E be a multi-normed space, and consider its associated ℓ^∞-norm $\|\cdot\|$ on $\ell^\infty \otimes E$. Since $\|\cdot\| \leq \|\cdot\|_\pi$, there is a dense-range contraction

$$(\ell^\infty \otimes E, \|\cdot\|_\pi) \to (\ell^\infty \otimes E, \|\cdot\|)\,.$$

The dual of this map is an injective contraction

$$(\ell^\infty \otimes E)' \hookrightarrow (\ell^\infty \widehat{\otimes} E)' = \mathcal{B}(\ell^\infty, E')\,,$$

and so we can identify $(\ell^\infty \otimes E)'$ with a subspace of $\mathcal{B}(\ell^\infty, E')$; this subspace is denoted by $\mathcal{B}_\beta(\ell^\infty, E')$, where β is the norm induced by $(\ell^\infty \otimes E)'$.

The space $\ell^1 \otimes E'$ acts linearly on $\ell^\infty \otimes E$ by the specification

$$\langle b \otimes x,\, a \otimes \lambda\rangle = \langle b, a\rangle \,\langle x, \lambda\rangle \qquad (a \otimes \lambda \in \ell^1 \otimes E',\; b \otimes x \in \ell^\infty \otimes E)\,.$$

Since the ℓ^∞-norm on $\ell^\infty \otimes E$ is reasonable, we see that the action of $\ell^1 \otimes E'$ on $\ell^\infty \otimes E$ is continuous. Thus we have a natural inclusion $\ell^1 \otimes E' \hookrightarrow (\ell^\infty \otimes E)'$.

Now consider the c_0-norm $\|\cdot\|$ on $c_0 \otimes E$ that is associated with the given multi-norm based on E. Then the natural inclusion $(c_0 \otimes E, \|\cdot\|) \hookrightarrow (\ell^\infty \otimes E, \|\cdot\|)$ is isometric, and therefore we have a natural quotient mapping $(\ell^\infty \otimes E)' \twoheadrightarrow (c_0 \otimes E)'$. We note that $\ell^1 \otimes E$ acts on $c_0 \otimes E$ by restricting its action on $\ell^\infty \otimes E$ to $c_0 \otimes E$, that this new action also induces an inclusion $\ell^1 \otimes E' \hookrightarrow (c_0 \otimes E)'$, and that the following diagram commutes:

$$\begin{tikzcd} \ell^1 \otimes E' \arrow[r, hook] \arrow[dr, hook] & (\ell^\infty \otimes E)' \arrow[d, two heads] \\ & (c_0 \otimes E)'\,. \end{tikzcd} \tag{7}$$

Theorem 4.2. *Let E be a multi-normed space. Then there exists a unique ℓ^1-norm on $\ell^1 \otimes E'$ such that the two inclusions in the diagram* (7) *are isometric. Furthermore, the image of $\ell^1 \otimes E'$ in $(\ell^\infty \otimes E)'$ is a norming set for $\ell^\infty \otimes E$.*

Proof. The uniqueness statement is obvious, and the only way to define this ℓ^1-norm on $\ell^1 \otimes E'$ is by taking the restriction of the norm of $(\ell^\infty \otimes E)'$. The fact that the norm on $\ell^1 \otimes E'$ just defined is an ℓ^1-norm follows directly from the properties of the ℓ^∞-norm on $\ell^\infty \otimes E$.

We shall now prove that the inclusion $\ell^1 \otimes E' \hookrightarrow (c_0 \otimes E)'$ is also isometric. Thus consider $\sigma = \sum_{i=1}^m a_i \otimes \lambda_i \in \ell^1 \otimes E'$, and take $\varepsilon > 0$. By the definition of the norm on $\ell^1 \otimes E'$, we see that there exists

$\tau = \sum_{j=1}^{n} b_j \otimes x_j \in \ell^\infty \otimes E$ with $\|\tau\| = 1$ and such that $\langle \tau, \sigma \rangle > \|\sigma\| - \varepsilon$. For each $k \in \mathbb{N}$, let $P_k \in \mathcal{B}(\ell^\infty)$ be the projection onto the first k coordinates. Then it is clear that

$$\|\sigma\| - \varepsilon < \langle \tau, \sigma \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} \langle a_i, b_j \rangle \langle x_j, \lambda_i \rangle = \lim_{k \to \infty} \langle (P_k \otimes I_E)(\tau), \sigma \rangle \;;$$

further, we note that $(P_k \otimes I_E)(\tau) \in c_0 \otimes E$ with $\|(P_k \otimes I_E)(\tau)\| \leq \|\tau\| = 1$. This proves that the inclusion map $\ell^1 \otimes E' \hookrightarrow (c_0 \otimes E)'$ is isometric.

For the last statement, take $\tau = \sum_{i=1}^{n} a_i \otimes x_i \in \ell^\infty \otimes E$. By the Hahn–Banach theorem, there exists $T \in \mathcal{B}_\beta(\ell^\infty, E')$ with $\beta(T) = 1$ and $\langle \tau, T \rangle = \|\tau\|$. Let F be the linear span of the set $\{a_1, \ldots, a_n\}$ in ℓ^∞, and take $\varepsilon > 0$. Arguing as in the proof of Theorem 3.4, we see that there exist bounded linear operators $R, S : \ell^\infty \to \ell^\infty$ such that the range of R is contained in ℓ_N^∞ for some natural number N, such that $\|S\|\|R\| < 1 + \varepsilon$, and such that $(SR)|F = I_F$; moreover, we can arrange that $R = U'$ for some $U \in \mathcal{B}(\ell^1)$. Then TSP_N belongs to $\ell^1 \otimes E'$ (considered canonically as a subspace of $\mathcal{F}(\ell^\infty, E') \subset \mathcal{B}_\beta(\ell^\infty, E')$), and so

$$TSR = TSP_N U' \in \ell^1 \otimes E' .$$

We see that $\beta(TSR) \leq \beta(T)\|SR\| < 1 + \varepsilon$ and that

$$\langle \tau, TSR \rangle = \langle \tau, T \rangle = \|\tau\| .$$

This holds true for each $\varepsilon > 0$, and so $\ell^1 \otimes E'$ is a norming set for $\ell^\infty \otimes E$, as claimed. $\square$

Analogously, when E is a dual multi-normed space, we see that $\ell^\infty \otimes E'$ acts on $\ell^1 \otimes E$ by an action which satisfies the condition that

$$\langle b \otimes x,\, a \otimes \lambda \rangle = \langle a, b \rangle \langle \lambda, x \rangle \quad (a \otimes \lambda \in \ell^\infty \otimes E',\ b \otimes x \in \ell^1 \otimes E) .$$

The proof of the following theorem is essentially the same as that of Theorem 4.2.

Theorem 4.3. *Let E be a dual multi-normed space. Then the above dual pairing gives an injection $\ell^\infty \otimes E' \to (\ell^1 \otimes E)'$ which induces a multi-norm structure on E'. Furthermore, $c_0 \otimes E'$ is a norming set for $\ell^1 \otimes E$.* $\square$

Suppose that E is a multi-normed space. Then the above two theorems give us an ℓ^1-norm on $\ell^1 \otimes E'$ and then a c_0-norm on $c_0 \otimes E''$. Thus we have the following (see also [6, Theorem 2.34], where the proof is given by a different argument).

Corollary 4.4. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, and induce a dual multi-norm structure based on E'. Using this, induce a multi-norm $(\|\cdot\|_n'' : n \in \mathbb{N})$ based on E''. Then, for each $n \in \mathbb{N}$, the restriction of $\|\cdot\|_n''$ to the canonical image of E^n in $(E'')^n$ is equal to $\|\cdot\|_n$.* $\square$

5 The (p, q)-multi-norm

Following [6, §4.1], we now introduce an important class of multi-norms. Let E be a normed space, and take p, q with $1 \leq p, q < \infty$. For each $n \in \mathbb{N}$ and each $\boldsymbol{x} = (x_1, \ldots, x_n) \in E^n$, we define

$$\|\boldsymbol{x}\|_n^{(p,q)} = \sup \left\{ \left(\sum_{i=1}^{n} |\langle x_i, \lambda_i \rangle|^q \right)^{1/q} : \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (E')^n,\ \mu_{p,n}(\boldsymbol{\lambda}) \leq 1 \right\} .$$

It is clear that $\|\cdot\|_n^{(p,q)}$ is a norm on E^n. As proved in [6, Theorem 4.1], in the case where $1 \leq p \leq q < \infty$, the sequence $(\|\cdot\|_n^{(p,q)} : n \in \mathbb{N})$ is a multi-norm based on E.

Definition 5.1. Let E be a normed space, and take p, q with $1 \leq p \leq q < \infty$. Then the multi-norm $(\|\cdot\|_n^{(p,q)} : n \in \mathbb{N})$ described above is the (p,q)-*multi-norm* over E.

A subset of E is (p,q)-*multi-bounded* if it is multi-bounded with respect to the (p,q)-multi-norm. The (p,q)-*multi-bound* of such a set B is denoted by $\mathrm{mb}_{p,q}(B)$.

Remark 5.2. By [6, Theorem 4.6] (*cf.* (4)), the $(1,1)$-multi-norm is just the maximum multi-norm based on E.

Lemma 5.3. *Let E be a normed space, and take p, q with $1 \leq p, q < \infty$. Then, for each $n \in \mathbb{N}$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (E')^n$, we have*

$$\|\boldsymbol{\lambda}\|_n^{(p,q)} = \sup \left\{ \left(\sum_{i=1}^n |\langle x_i, \lambda_i \rangle|^q \right)^{1/q} : \boldsymbol{x} = (x_1, \ldots, x_n) \in E^n,\ \mu_{p,n}(\boldsymbol{x}) \leq 1 \right\}.$$

Proof. This is proved in [6, Proposition 4.10]; it follows from the Principal of Local Reflexivity. □

This lemma implies that, for each normed space E, the (p,q)-multi-norm based on E is the same as the one induced from the (p,q)-multi-norm based on E''.

Suppose now that E is a normed space and that p, q satisfy $1 \leq p \leq q < \infty$.

Definition 5.4. We denote by $\mathcal{B}_{p,q}(\ell^1, E)$ the subset of $\mathcal{B}(\ell^1, E)$ consisting of those operators T with the property that $\{T(\delta_k) \colon k \in \mathbb{N}\}$ is (p,q)-multi-bounded in E. We define the norm on $\mathcal{B}_{p,q}(\ell^1, E)$ by

$$\alpha_{p,q}(T) := \mathrm{mb}_{p,q} \{T(\delta_k) \colon k \in \mathbb{N}\}.$$

By Proposition 2.9, we see that $\mathcal{B}_{p,q}(\ell^1, E) = \mathcal{M}(\ell^1, E)$ when ℓ^1 is given the minimum multi-norm and E is given the (p,q)-multi-norm; moreover,

$$\alpha_{p,q}(T) = \|T\|_{mb} \quad (T \in \mathcal{B}_{p,q}(\ell^1, E)).$$

In particular, it follows that

$$\mathcal{F}(\ell^1, E) \subset \mathcal{B}_{p,q}(\ell^1, E) \subset \mathcal{B}(\ell^1, E),$$

and that, indeed, $(\mathcal{B}_{p,q}(\ell^1, E), \alpha_{p,q})$ is a normed space; it is a Banach space when E is a Banach space.

From the discussion above, we see also that the natural injection from $\ell^\infty \otimes E$ into $(\mathcal{B}_{p,q}(\ell^1, E), \alpha_{p,q})$ is isometric with respect to the ℓ^∞-norm on $\ell^\infty \otimes E$ associated with the (p,q)-multi-norm.

Recall from [8, Chapter 10] that an operator T from a normed space E into another normed space F is (q,p)-*summing* if there exists a constant C such that

$$\left(\sum_{i=1}^n \|Tx_i\|^q \right)^{1/q} \leq C\, \mu_{p,n}(x_1, \ldots, x_n) \quad (x_1, \ldots, x_n \in E,\ n \in \mathbb{N}).$$

The smallest such constant C is denoted by $\pi_{q,p}(T)$. The set of (q,p)-summing operators, denoted by $\Pi_{q,p}(E, F)$, is a normed space when equipped with the norm $\pi_{q,p}$; it is a Banach space when E and F are Banach spaces. When $p = q$, we shall write $\mathcal{B}_p$, Π_p, and π_p instead of $\mathcal{B}_{p,p}$, $\Pi_{p,p}$, and $\pi_{p,p}$, respectively. The space (Π_p, π_p) consisting of all p-summing operators has been studied by many authors; see [7], [8], and [13], for example.

Proposition 5.5. *Let E be normed space, and take p, q with $1 \leq p \leq q < \infty$. Suppose that $T \in \mathcal{B}(\ell^1, E)$. Then $T \in \mathcal{B}_{p,q}(\ell^1, E)$ if and only if $T' \in \Pi_{q,p}(E', \ell^\infty)$. In this case, we have $\alpha_{p,q}(T) = \pi_{q,p}(T')$.*

Proof. Suppose that $T \in \mathcal{B}_{p,q}(\ell^1, E)$. From the previous discussion, $\alpha_{p,q}(T) = \mathrm{mb}_{p,q} T(\ell^1_{[1]})$, and so $\alpha_{p,q}(T)$ is the smallest constant C such that

$$C\,\mu_{p,n}(\lambda_1, \dots, \lambda_n) \geq \left(\sum_{i=1}^{n} |\langle Ta_i, \lambda_i\rangle|^q\right)^{1/q} = \left(\sum_{i=1}^{n} |\langle a_i, T'\lambda_i\rangle|^q\right)^{1/q}$$

for every $n \in \mathbb{N}$, $a_1, \dots, a_n \in \ell^1_{[1]}$, and $\lambda_1, \dots, \lambda_n \in E'$. Taking the supremum over all elements $a_1, \dots, a_n \in \ell^1_{[1]}$, we see that $\alpha_{p,q}(T)$ is the smallest constant C such that

$$C\,\mu_{p,n}(\lambda_1, \dots, \lambda_n) \geq \left(\sum_{i=1}^{n} \|T'\lambda_i\|^q\right)^{1/q}.$$

Thus $T' \in \Pi_{q,p}(E', \ell^\infty)$ and $\pi_{q,p}(T') = \alpha_{p,q}(T)$.

The converse follows in the same way. ☐

In the following result, we shall use the fact that $L^1_{\mathbb{R}}(\Omega)$ is an AL-space as a (real) Banach lattice, and so its dual space is an AM-space with an order-unit. Thus there is a compact space K and a linear isometry $\theta : L^1(\Omega)' \to C(K)$ such that $\theta | L^1_{\mathbb{R}}(\Omega)'$ is an order-isomorphism from $L^1_{\mathbb{R}}(\Omega)'$ onto $C_{\mathbb{R}}(K)$: this is Kakutani's representation theorem. For these results on Banach lattices, see [1, §12], for example.

Theorem 5.6. *Let (Ω, μ) be a measure space, and take p, q, r with $1 \leq p < q < r < \infty$. Then the (p,q)-multi-norm is equivalent to the $(1,q)$-multi-norm based on $L^1(\Omega)$ and dominates the (r,r)-multi-norm based on $L^1(\Omega)$.*

Proof. By [8, Theorem 10.9], we have

$$\Pi_{q,p}(C(K), \ell^\infty) = \Pi_{q,1}(C(K), \ell^\infty) \subset \Pi_r(C(K), \ell^\infty)$$

for each compact space K, where the last inclusion is continuous. The conclusion then follows from Proposition 5.5. ☐

We shall consider the mutual equivalence of various (p,q)-multi-norms and some other multi-norms based on certain Banach spaces in [3].

Theorem 5.7. *Let E be a Banach space and take $p \in [1, \infty)$. Then every (p,p)-multi-bounded subset of E is relatively weakly compact.*

Proof. Let $T \in \mathcal{M}(\ell^1, E) = \mathcal{B}_p(\ell^1, E)$. By Proposition 5.5, $T' \in \Pi_p(E', \ell^\infty)$. By the Pietsch Factorization Theorem, every p-summing operator is weakly compact [8, Theorem 2.17]. It follows that T' is weakly compact, and hence so is T. By Proposition 3.10, every (p,p)-multi-bounded subset of E must be relatively weakly compact. ☐

Corollary 5.8. *Let (Ω, μ) be a measure space, and take p, q with $1 \leq p \leq q < \infty$. Then every (p,q)-multi-bounded subset of $L^1(\Omega)$ is relatively weakly compact.*

Proof. This is a consequence of Theorems 5.6 and 5.7. ☐

Remark 5.9. Theorem 5.7 cannot be generalized to (p,q)-multi-bounded sets in the case where $p < q$. Indeed, it is a result of Kwapień and Pełczyński that

$$S : (\alpha_n) \mapsto \left(\sum_{i=1}^{n} \alpha_i\right)_{n=1}^{\infty}, \quad \ell^1 \to \ell^\infty,$$

is (q,p)-summing for every $1 \leq p < q < \infty$, but S is not weakly compact (*cf.* [8, p. 210]). Consider the operator $T \in \mathcal{B}(\ell^1, c_0)$ defined by requiring that

$$T(\delta_n) = \sum_{i=1}^{n} \delta_i \quad (n \in \mathbb{N}) .$$

Then $T' = S$. In particular, T is not weakly compact, and so, by the Kreĭn-Šmulian theorem [16, Theorem 2.8.14], the set $\{T(\delta_n) : n \in \mathbb{N}\}$ is not relatively weakly compact. However, it follows from Proposition 5.5 that $T \in \mathcal{B}_{p,q}(\ell^1, c_0)$, and so $\{T(\delta_n) : n \in \mathbb{N}\}$ is (p,q)-multi-bounded. Thus we obtain a subset of c_0 which is (p,q)-multi-bounded for every $1 \leq p < q < \infty$, but which is not relatively weakly compact.

6 The standard q-multi-norm on L^p spaces

Let (Ω, μ) be a measure space, and take p, q with $1 \leq p \leq q < \infty$. In [6, §4.2], there is a definition and discussion of the *standard q-multi-norm* on $E := L^p(\Omega)$. We recall the definition.

Take $n \in \mathbb{N}$. For each partition $\boldsymbol{X} = \{X_1, \dots, X_n\}$ of Ω into measurable subsets and each $f_1, \dots, f_n \in L^p(\Omega)$, we define

$$\|(f_1, \dots, f_n)\|_n^{[q]} = \sup_{\boldsymbol{X}} \Big(\sum_{i=1}^{n} \|P_{X_i} f_i\|^q \Big)^{1/q} .$$

Here $P_{X_i} : f \mapsto f\chi_{X_i}$ is the projection of $L^p(\Omega)$ onto $L^p(X_i)$, $\| \cdot \|$ is the L^p-norm, and the supremum is taken over all such measurable partitions $\boldsymbol{X}$ of Ω. It is verified in [6, §4.2.1] that $(\| \cdot \|_n^{[q]} : n \in \mathbb{N})$ is a multi-norm based on $L^p(\Omega)$; it is called the standard q-multi-norm on $L^p(\Omega)$ in [6, Definition 4.21].

In terms of tensor norms, we have the following, which applies in the special case where $q = p$.

Theorem 6.1. *Let Ω be a measure space, and take $p \geq 1$. Then the standard p-multi-norm induces the c_0-norm on $c_0 \otimes L^p(\Omega)$ which comes from identifying $c_0 \otimes L^p(\Omega)$ with a subspace of the vector-valued space $L^p(\Omega, c_0)$.*

Proof. We set $F = L^p(\Omega, c_0)$. Take $n \in \mathbb{N}$, and let $f_1, \dots, f_n \in L^p(\Omega)$, so that

$$\Big\| \sum_{i=1}^{n} \delta_i \otimes f_i \Big\|_F^p = \int_\Omega \Big\| \sum_{i=1}^{n} \delta_i \otimes f_i(t) \Big\|_{c_0}^p \, \mathrm{d}m(t) = \int_\Omega \max_{i \in \mathbb{N}_n} |f_i(t)|^p \, \mathrm{d}m(t) .$$

For each $i \in \mathbb{N}_n$, let Y_i be the set of points of Ω at which $|f_i|$ equals $\max\{|f_j| : j \in \mathbb{N}_n\}$. Set $X_1 = Y_1$ and $X_j = Y_j \setminus \bigcup_{i=1}^{j-1} Y_i$ for each $j \in \{2, \dots, n\}$, so that $\{X_1, \dots, X_n\}$ is a measurable partition of Ω. Then we see that

$$\sum_{i=1}^{n} \|\chi_{X_i} f_i\|^p = \sum_{i=1}^{n} \int_{X_i} |f_i(t)|^p \, \mathrm{d}m(t) = \int_\Omega \max_{i \in \mathbb{N}_n} |f_i(t)|^p \, \mathrm{d}m(t).$$

Thus $\|(f_1, \dots, f_n)\|_n^{[p]} \geq \| \sum_{i=1}^{n} \delta_i \otimes f_i \|_F$.

On the other hand, for each measurable partition $\boldsymbol{X} = \{X_1, \dots, X_n\}$ of Ω, we have

$$\sum_{i=1}^{n} \|\chi_{X_i} f_i\|^p = \sum_{i=1}^{n} \int_{X_i} |f_i(t)|^p \, \mathrm{d}m(t) \leq \int_\Omega \max_{i \in \mathbb{N}_n} |f_i(t)|^p \, \mathrm{d}m(t) ,$$

and so $\|(f_1, \dots, f_n)\|_n^{[p]} \leq \| \sum_{i=1}^{n} \delta_i \otimes f_i \|_F$.

Thus $\|(f_1, \dots, f_n)\|_n^{[p]} = \| \sum_{i=1}^{n} \delta_i \otimes f_i \|_F$, and so the result follows. □

From the above theorem, it follows that, for every $f_1, \dots, f_n \in L^p(\Omega)$, we have

$$\|(f_1, \dots, f_n)\|_n^{[p]} = \|\,|f_1| \vee \cdots \vee |f_n|\,\| = \|(f_1, \dots, f_n)\|_n^L .$$

We do not have a similar description of the standard q-multi-norm on $L^p(\Omega)$ when $q > p$.

When $p = 1$, it is well-known that $L^1(\Omega) \widehat{\otimes} E = L^1(\Omega, E)$ for any Banach space E, and so the standard 1-multi-norm on $L^1(\Omega)$ is the maximum multi-norm (*cf.* [6, Theorem 4.23]). Thus, for $f_1, \dots, f_n \in L^1(\Omega)$, we have

$$\|(f_1, \dots, f_n)\|_n^{\max} = \|\,|f_1| \vee \cdots \vee |f_n|\,\| = \|(f_1, \dots, f_n)\|_n^{[1]} = \|(f_1, \dots, f_n)\|_n^L . \tag{8}$$

7 The extension of the standard q-multi-norm

In this section, we shall give another description of the (p, q)-multi-norm based on a space E; the description will be required for our main theorem in §9. In that later section, we shall need to prove the amenability of a locally compact group G by using information about $\mathcal{B}(L^1(G), L^p(G))$; the latter information is provided directly by the injectivity of $L^p(G)$. The main result of this section will give us a necessary bridge between G and $\mathcal{B}(L^1(G), L^p(G))$.

Let E be a Banach space, and let $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space with $F \neq \{0\}$. For each $n \in \mathbb{N}$, we define a norm $\|\cdot\|_n^F$ on the space E^n by setting

$$\|(x_1, \dots, x_n)\|_n^F = \sup\left\{\|(Tx_1, \dots, Tx_n)\|_n : T \in \mathcal{B}(E, F)_{[1]}\right\} \quad (x_1, \dots, x_n \in E) .$$

It is immediately checked that $(\|\cdot\|_n^F : n \in \mathbb{N})$ is a multi-norm based on E and that

$$\mathcal{M}(E, F) = \mathcal{B}(E, F) \quad \text{with} \quad \|T\|_{mb} = \|T\| \quad (T \in \mathcal{M}(E, F))$$

when $\mathcal{M}(E, F)$ is calculated with respect to the multi-norm $(\|\cdot\|_n^F : n \in \mathbb{N})$ based on E.

Now suppose that $(|||\cdot|||_n : n \in \mathbb{N})$ is a multi-norm based on E with the property that, with respect to this new multi-norm, we have

$$\mathcal{M}(E, F) = \mathcal{B}(E, F) \quad \text{with} \quad \|T\|_{mb} = \|T\| \quad \text{for each } T \in \mathcal{M}(E, F) . \tag{9}$$

Then we see that

$$\begin{aligned} \|\boldsymbol{x}\|_n^F &= \sup\left\{\|(Tx_1, \dots, Tx_n)\|_n : T \in \mathcal{B}(E, F)_{[1]}\right\} \\ &= \sup\left\{\|(Tx_1, \dots, Tx_n)\|_n : T \in \mathcal{M}(E, F)_{[1]}\right\} \leq |||\boldsymbol{x}|||_n \end{aligned}$$

for each $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$ and $n \in \mathbb{N}$. Thus $(\|\cdot\|_n^F : n \in \mathbb{N})$ is the minimum multi-norm in $\mathcal{E}_E$ satisfying condition (9).

Definition 7.1. The multi-norm $(\|\cdot\|_n^F : n \in \mathbb{N})$ described above is the *extension* to E of the multi-norm on F.

There is a discussion of extensions of multi-norms in [6, §6.5].

Now, let (Ω, μ) be a measure space, and take p, q with $1 \leq p \leq q < \infty$. For the rest of this section, we shall suppose also that $L^p(\Omega)$ is *infinite–dimensional*. This is the same as requiring that, for every $n \in \mathbb{N}$, there exist pairwise–disjoint, measurable subsets $X_1, \dots, X_n$ of Ω such that $0 < \mu(X_i) < \infty$ for all $i \in \mathbb{N}_n$.

Again, we write p' for the conjugate index of p, and set $F = L^{p'}(\Omega)$. For each $n \in \mathbb{N}$, let D_n be the set of elements $(\lambda_1, \dots, \lambda_n) \in (F_{[1]})^n$ such that the subsets $\operatorname{supp} \lambda_1, \dots, \operatorname{supp} \lambda_n$ of Ω are pairwise disjoint. Then it is immediate from the definition of the standard q-multi-norm on $L^p(\Omega)$ that

$$\|(f_1, \dots, f_n)\|_n^{[q]} = \sup\left\{\left(\sum_{i=1}^n |\langle f_i, \lambda_i\rangle|^q\right)^{1/q} : (\lambda_1, \dots, \lambda_n) \in D_n\right\} \tag{10}$$

for every $(f_1, \ldots, f_n) \in L^p(\Omega)^n$ and every $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, set

$$B_n(E') = \left\{(T'\varphi_1, \ldots, T'\varphi_n) \colon T \in \mathcal{B}(E, L^p(\Omega))_{[1]},\ (\varphi_1, \ldots, \varphi_n) \in D_n\right\} \subset (E')^n .$$

Lemma 7.2. *Let E be a Banach space. Then $B_n(E') = \{\boldsymbol{\lambda} \in (E')^n : \mu_{p,n}(\boldsymbol{\lambda}) \leq 1\}$ for each $n \in \mathbb{N}$.*

Proof. Set $C_n(E') = \{\boldsymbol{\lambda} \in (E')^n : \mu_{p,n}(\boldsymbol{\lambda}) \leq 1\}$.

Let $T \in \mathcal{B}(E, L^p(\Omega))_{[1]}$ and $(\varphi_1, \ldots, \varphi_n) \in D_n$, so that $T' : F \to E'$. For each $i \in \mathbb{N}_n$, set $X_i = \operatorname{supp} \varphi_i$, and then set $\boldsymbol{\lambda} = (T'\varphi_1, \ldots, T'\varphi_n) \in (E')^n$. For each $x \in E_{[1]}$, we have

$$\left(\sum_{i=1}^n \left|\langle T'\varphi_i, x\rangle\right|^p\right)^{1/p} = \left(\sum_{i=1}^n |\langle \varphi_i, Tx\rangle|^p\right)^{1/p} \leq \left(\sum_{i=1}^n \|\chi_{X_i} Tx\|^p\right)^{1/p} \leq \|Tx\| \leq 1 .$$

Hence $\mu_{p,n}(\boldsymbol{\lambda}) \leq 1$, and so $B_n(E') \subset C_n(E')$.

Conversely, let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in C_n(E')$, and then choose pairwise–disjoint, measurable subsets $X_1, \ldots, X_n$ of Ω with $0 < \mu(X_i) < \infty$ $(i \in \mathbb{N}_n)$. Set

$$\varphi_i = \begin{cases} \dfrac{\chi_{X_i}}{\mu(X_i)^{1/p'}} & \text{when } p > 1 \\ \chi_{X_i} & \text{when } p = 1 \end{cases} \qquad (i \in \mathbb{N}_n) ,$$

so that $(\varphi_1, \ldots, \varphi_n) \in D_n$. Next set

$$T = \sum_{i=1}^n \lambda_i \otimes \frac{\chi_{X_i}}{\mu(X_i)^{1/p}} \in E' \otimes L^p(\Omega) \subset \mathcal{B}(E, L^p(\Omega)) ,$$

where we again use the identification of (1). For $x \in E$, we have

$$\|Tx\| = \left\|\sum_{i=1}^n \langle x, \lambda_i\rangle \frac{\chi_{X_i}}{\mu(X_i)^{1/p}}\right\| = \left(\sum_{i=1}^n |\langle x, \lambda_i\rangle|^p\right)^{1/p} \leq \mu_{p,n}(\boldsymbol{\lambda})\|x\| \leq \|x\| .$$

It follows that $T \in \mathcal{B}(E, L^p(\Omega))_{[1]}$. Since it can be seen that $\boldsymbol{\lambda} = (T'\varphi_1, \ldots, T'\varphi_n)$, we have $C_n(E') \subset B_n(E')$.

Hence $B_n(E') = C_n(E')$, as required. $\square$

The following proposition now follows from Lemma 7.2 and equation (10).

Proposition 7.3. *Let E be a Banach space, and take p, q with $1 \leq p \leq q < \infty$. Let Ω be a measure space such that $L^p(\Omega)$ is infinite–dimensional. Then the extension to E of the standard q-multi-norm on $L^p(\Omega)$ is the (p, q)-multi-norm on E.* $\square$

When $p > 1$, since $L^p(\Omega)'' = L^p(\Omega)$, we can do the same as the above for E''.

Proposition 7.4. *Let E be a Banach space, and take p, q with $1 < p \leq q < \infty$. Let Ω be a measure space such that $L^p(\Omega)$ is infinite–dimensional. Then, for each $\Phi_1, \ldots, \Phi_n \in E''$, we have*

$$\|(\Phi_1, \ldots, \Phi_n)\|_n^{(p,q)} = \sup \|(T''(\Phi_1), \ldots, T''(\Phi_n))\|_n^{[q]}$$

where the supremum is taken over all $T \in \mathcal{B}(E, L^p(\Omega))_{[1]}$.

Proof. Let $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_n) \in (E'')^n$. By Lemma 5.3, we have

$$\|\mathbf{\Phi}\|_n^{(p,q)} = \sup\left\{\left(\sum_{i=1}^n |\langle \Phi_i, \lambda_i\rangle|^q\right)^{1/q} : \boldsymbol{\lambda} \in (E')^n,\ \mu_{p,n}(\boldsymbol{\lambda}) \leq 1\right\}.$$

By Lemma 7.2, this is equal to

$$\|\Phi\|_n^{(p,q)} = \sup\left\{\left(\sum_{i=1}^n \left|\left\langle \Phi_i, T'\varphi_i\right\rangle\right|^q\right)^{1/q} : T \in \mathcal{B}(E, L^p(\Omega))_{[1]},\ (\varphi_1, \ldots, \varphi_n) \in D_n\right\}.$$

Hence

$$\begin{aligned}\|\Phi\|_n^{(p,q)} &= \sup\left\{\left(\sum_{i=1}^n \left|\left\langle T''\Phi_i, \varphi_i\right\rangle\right|^q\right)^{1/q} : T \in \mathcal{B}(E, L^p(\Omega))_{[1]},\ (\varphi_1, \ldots, \varphi_n) \in D_n\right\} \\ &= \sup\left\{\|(T''(\Phi_1), \ldots, T''(\Phi_n))\|_n^{[q]} : T \in \mathcal{B}(E, L^p(\Omega))_{[1]}\right\}\end{aligned}$$

by equation (10), which gives the result. □

8 Left (p, q)-multi-invariant means

In this section, we shall generalize the concept of a left-invariant mean for a locally compact group.

Definition 8.1. Let G be a locally compact group, and take p, q with $1 \leq p \leq q < \infty$. A functional $\Lambda \in L^\infty(G)'$ is *left (p, q)-multi-invariant* if the set $\{s \cdot \Lambda \colon s \in G\}$ is multi-bounded in the (p, q)-multi-norm. The group G is *left (p, q)-amenable* if there exists a left (p, q)-multi-invariant mean on $L^\infty(G)$.

The idea behind this definition is to attempt to measure the 'left-invariance' of a mean $\Lambda \in L^\infty(G)'$ by measuring the growth of the sets $\{s \cdot \Lambda : s \in F\}$ as F ranges through all the finite subsets of G.

It follows immediately from the multi-norm axiom (A4) and Theorem 5.6 that we have the following implications for a mean $\Lambda \in L^\infty(G)'$: for every $1 \leq p < q < r < \infty$, we have

$$\begin{array}{ccc} \text{left-invariant} \Rightarrow \text{left } (q,q)\text{-invariant} \Rightarrow & \text{left } (p,q)\text{-invariant} & \\ & \Updownarrow & \\ & \text{left } (1,q)\text{-invariant} & \Rightarrow \text{left } (r,r)\text{-invariant}\,. \end{array}$$

We shall now show that the left (p, q)-multi-invariance property is preserved when passing from a functional on $L^\infty(G)$ to an appropriate mean; this is analogous to a standard property of left-invariant functionals.

Lemma 8.2. *Let G be a locally compact group, and take p, q with $1 \leq p \leq q < \infty$. Suppose that Λ is a non-zero, left (p, q)-multi-invariant functional on $L^\infty(G)$. Then $|\Lambda| / \|\Lambda\|$ is a left (p, q)-invariant mean on $L^\infty(G)$, and G is left (p, q)-amenable.*

Proof. Recall that $L^\infty(G)'$ can be identified isometrically as a Banach lattice with $L^1(\Omega)$ for some measure space (Ω, μ). Set $\widetilde{\Lambda} := |\Lambda| / \|\Lambda\|$. Since $\|\,|\Lambda|\,\| = \|\Lambda\| = \langle 1, |\Lambda|\rangle$, it is clear that $\widetilde{\Lambda}$ is a mean on $L^\infty(G)$. Since $\mu_{p,n}(\varphi_1, \ldots, \varphi_n) = \mu_{p,n}(\psi_1, \ldots, \psi_n)$ for every $n \in \mathbb{N}$ and every $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n \in L^\infty(\Omega)$ with $|\varphi_i| = |\psi_i|$ $(i \in \mathbb{N}_n)$ [13, 2.6], we see that

$$\|(\Lambda_1, \ldots, \Lambda_n)\|_n^{(p,q)} = \|(|\Lambda_1|, \ldots, |\Lambda_n|)\|_n^{(p,q)} \qquad (\Lambda_1, \ldots, \Lambda_n \in L^\infty(G)')\,.$$

Now note that $|s \cdot \Lambda| = s \cdot |\Lambda|$ for every $s \in G$, and so $\{s \cdot \widetilde{\Lambda} \colon s \in G\}$ is multi-bounded in the (p, q)-multi-norm. The result follows. □

It turns out that left (p,q)-amenability is the same as amenability for a locally compact group G, as Theorem 8.4, given below, will show. We shall use the Ryll-Nardzewski fixed point theorem; to be explicit, we first quote a special form of the version of this theorem given in [9, Theorem A.2.2] and [17, §2.36].

Theorem 8.3. *Let E be a Banach space, and let K be a convex, weakly compact subset of E. Suppose that Σ is a semigroup of affine maps from K to K such that $\|Tx - Ty\| = \|x - y\|$ for all $x, y \in K$ and $T \in \Sigma$. Then there exists $x_0 \in K$ such that $Tx_0 = x_0$ for each $T \in \Sigma$.* □

Theorem 8.4. *Let G be a locally compact group, and take p, q with $1 \leq p \leq q < \infty$. Then G is amenable if and only if G is left (p,q)-amenable.*

Proof. We need to prove only the 'if' part.

So, suppose that Λ is a left (p,q)-invariant mean on $L^\infty(G)$; that is $\{s \cdot \Lambda \colon s \in G\}$ is (p,q)-multi-bounded in $L^\infty(G)'$. By Corollary 5.8 and either Lemma 2.7 or the Kreĭn-Šmulian theorem, the closed convex hull K of $\{s \cdot \Lambda \colon s \in G\}$ is weakly compact. For each $s \in G$, consider the map $L_s : \Psi \mapsto s \cdot \Psi$, $K \to K$. We obtain a group $\Sigma := \{L_s \colon s \in G\}$ of isometric affine maps. By Theorem 8.3, there exists $\Lambda_0 \in K$ which is a common fixed point for the set $\{L_s : \ s \in G\}$. Obviously, Λ_0 must be a left-invariant mean on $L^\infty(G)$. Hence the group G is amenable. □

Remark 8.5. When $L^\infty(G)$ has a left $(1,1)$-multi-invariant mean Λ, a left-invariant mean on $L^\infty(G)$ can be explicitly constructed, as follows. Consider Λ as an element of the real Banach lattice $L^\infty_{\mathbb{R}}(G)'$. For each finite subset $F = \{s_1, \dots, s_n\}$ of G, we set

$$\Psi_F := (s_1 \cdot \Lambda) \vee \cdots \vee (s_n \cdot \Lambda)\,.$$

Then we have an upward-directed net of positive linear functionals on $L^\infty(G)$. This net is bounded because Λ is left $(1,1)$-multi-invariant, where we use equation (8) (*cf.* Remark 5.2), and so its weak* limit Ψ exists and Ψ must be the supremum of $\{s \cdot \Lambda \colon s \in G\}$. It follows that Ψ is left-invariant.

Remark 8.6. There is an obvious definition of a *right (p,q)-multi-invariant mean.* Set $A = L^1(G)$, and let $\Lambda \in A''$ be a left (p,q)-multi-invariant mean. Define $\theta : A \to A$ by

$$\theta(a)(s) = a(s^{-1})\Delta(s^{-1}) \quad (a \in A, s \in G)\,.$$

Then $\theta'' : A'' \to A''$ takes the set $\{s \cdot \Lambda : s \in G\}$ to the set $\{\theta''(\Lambda) \cdot s : s \in G\}$, and $\theta'(1) = 1$. Since θ'' automatically belongs to $\mathcal{M}(A'', A'')$, it follows that $\theta''(\Lambda)$ is a right (p,q)-multi-invariant mean on G.

9 Injectivity and flatness of the module $L^p(G)$

Let G be a locally compact group, and take $p \in (1,\infty)$. In this section, we shall give an answer to the question of when $L^p(G)$ is injective and when it is flat in $L^1(G)$-**mod**. Now we shall write $\|\cdot\|_p$ for the norm on $L^p(G)$; we take q to be the conjugate index to p.

First, we shall prove that the injectivity of $L^p(G)$ in $L^1(G)$-**mod** implies the amenability of G. For this, we shall use a coretraction problem to show that, in the case where $L^p(G)$ is injective, $L^\infty(G)$ must have a left (p,p)-multi-invariant mean, and then we shall apply the result from the previous section.

We set $J = \mathcal{B}(L^1(G), L^p(G))$. We now define an action of G on the space J by

$$(t * U)\,(a) = t \cdot U(t^{-1} \cdot a) \quad (a \in L^1(G))$$

for each $U \in J$ and $t \in G$. For each $U \in J$ and $a \in L^1(G)$, the map $t \mapsto (t * U)(a)$, $G \to L^p(G)$, is continuous; this follows from the inequality

$$\begin{aligned}
\left\|t \cdot U(t^{-1} \cdot a) - U(a)\right\|_p &\leq \left\|t \cdot U(t^{-1} \cdot a) - t \cdot U(a)\right\|_p + \|t \cdot U(a) - U(a)\|_p \\
&= \left\|U(t^{-1} \cdot a - a)\right\|_p + \|t \cdot U(a) - U(a)\|_p \\
&\leq \|U\|\|t^{-1} \cdot a - a\|_1 + \|t \cdot U(a) - U(a)\|_p
\end{aligned}$$

and the continuity of translation in $L^1(G)$ and $L^p(G)$ [2, Proposition 3.3.11].

Proposition 9.1. *There is a Banach left $L^1(G)$-module structure on J given by a product $*$, where*

$$(b * U)(a) = \int_G b(t)\,(t * U)(a)\,\mathrm{d}m(t) \quad (a, b \in L^1(G),\, U \in J)\,. \tag{11}$$

Proof. This is similar to the proof that $L^p(G)$ is a left $L^1(G)$-module [2, Theorem 3.3.19].

Fix $U \in J$ and $a, b \in L^1(G)$, and let $\psi \in C_{00}(G)$. By Hölder's inequality, we have

$$\int_G \left|U(t^{-1} \cdot a)(t^{-1}s)\right| |\psi(s)|\ \mathrm{d}m(s) \le \|t \cdot U(t^{-1} \cdot a)\|_p \|\psi\|_q \le \|U\| \|a\|_1 \|\psi\|_q$$

for each $t \in G$. Now define $\Lambda(\psi)$ for $\psi \in C_{00}(G)$ by

$$\begin{aligned}\Lambda(\psi) &= \int_G (b * U)(a)(s)\psi(s)\,\mathrm{d}m(s)\\ &= \int_G \left(\int_G b(t)U(t^{-1} \cdot a)(t^{-1}s)\,\mathrm{d}m(t)\right)\psi(s)\,\mathrm{d}m(s)\\ &= \int_G b(t)\left(\int_G U(t^{-1} \cdot a)(t^{-1}s)\psi(s)\,\mathrm{d}m(s)\right)\,\mathrm{d}m(t)\,.\end{aligned}$$

Then

$$|\Lambda(\psi)| \le \|b\|_1 \|U\| \|a\|_1 \|\psi\|_q\,,$$

and so Λ extends to an element of $L^q(G)'$ of norm at most $\|b\|_1\|U\|\|a\|_1$. Hence, by the identification of $L^q(G)'$ with $L^p(G)$, we see that $(b * U)(a) \in L^p(G)$ with

$$\|(b * U)(a)\| \le \|b\|_1 \|U\| \|a\|_1\,,$$

and so $b * U \in J$ with $\|b * U\| \le \|b\|_1\|U\|$.

The associativity formula $a * (b * U) = (a * b) * U$ holds for $a, b \in C_{00}(G)$ and $U \in J$, and so it holds for all $a, b \in L^1(G)$ because $C_{00}(G)$ is dense in $L^1(G)$. □

We shall denote the above left $L^1(G)$-module by $\widetilde{J} = (J, *)$. (We could similarly define a right multiplication on J such that J becomes a Banach $L^1(G)$-bimodule.)

Now we define an embedding $\widetilde{\Pi} : L^p(G) \to \widetilde{J}$ by

$$(\widetilde{\Pi}x)(a) = \varphi_G(a)x \quad (a \in L^1(G))\,,$$

where $x \in L^p(G)$ and φ_G is the augmentation character on $L^1(G)$. Certainly $\widetilde{\Pi} \in \mathcal{B}(L^p(G), \widetilde{J})$. For each $b \in L^1(G)$, we have

$$\left(b * \widetilde{\Pi}x\right)(a) = \int_G b(t)\varphi_G(t^{-1} \cdot a)\, t \cdot x\,\mathrm{d}m(t) = \varphi_G(a) b \star x = \widetilde{\Pi}(b \star x)(a) \quad (a \in L^1(G))\,,$$

and so $\widetilde{\Pi}$ is a left $L^1(G)$-module morphism; further, $\widetilde{\Pi}$ is admissible (a left inverse of $\widetilde{\Pi}$ in the category of Banach spaces is the map $U \mapsto U(a_0)$ for any $a_0 \in L^1(G)$ with $\varphi_G(a_0) = 1$).

Proposition 9.2. *Let G be a locally compact group, and take $p \in (1, \infty)$. Suppose that $L^p(G)$ is injective in $L^1(G)$-**mod**. Then the morphism $\widetilde{\Pi}$ is a coretraction in $L^1(G)$-**mod**.*

Proof. This follows immediately from the definition of injectivity. □

The converse of the above proposition is also true, as can be seen from the proof of Theorem 9.6, below.

We shall need the following generalization of [5, Lemma 5.2].

In the next three results, we suppose that Ω is a measure space, and take $p \in (1, \infty)$; the norm on $L^p(\Omega)$ is $\|\cdot\|_p$.

For $n \in \mathbb{N}$, we set $D_n = \{-1, 1\}^n$, and, for $j \in \mathbb{N}_n$, we set

$$D_n^+(j) = \{(d_1, \dots, d_n) \in D_n : d_j = 1\}\,, \quad D_n^-(j) = \{(d_1, \dots, d_n) \in D_n : d_j = -1\}\,.$$

Lemma 9.3. *Let $n \in \mathbb{N}$, and suppose that $F : \mathbb{N}_n \times \mathbb{N}_n \to L^p(\Omega)$ is a function. Set*

$$C = \max\left\{ \left(\sum_{j=1}^{n} \left\| \sum_{i=1}^{n} d_i F(i,j) \right\|_p^p \right)^{1/p} : (d_1, \dots, d_n) \in D_n \right\}\,.$$

Then $\sum_{j=1}^{n} \|F(j,j)\|_p^p \le C^p$.

Proof. Let $d = (d_1, \dots, d_n) \in D_n$, and set $x_{j,d} = \sum_{i=1}^{n} d_i F(i,j) \quad (j \in \mathbb{N}_n)$. By hypothesis, we have $\sum_{j=1}^{n} \|x_{j,d}\|_p^p \le C^p$. Since there are 2^n elements in D_n, we have

$$\sum_{j=1}^{n} \sum_{d \in D_n} \|x_{j,d}\|_p^p \le 2^n C^p\,.$$

For each $j \in \mathbb{N}_n$, write M_j for the set of the maps from $\mathbb{N}_n \setminus \{j\}$ to $\{-1, 1\}$. Then we can write the term $\sum_{d \in D_n} \|x_{j,d}\|_p^p$ as

$$\begin{aligned}
\sum_{d \in D_n} \|x_{j,d}\|_p^p &= \sum_{d \in D_n^+(j)} \|x_{j,d}\|_p^p + \sum_{d \in D_n^-(j)} \|x_{j,d}\|_p^p \\
&= \sum_{d \in M_j} \left(\left\| \sum_{i \ne j} d_i F(i,j) + F(j,j) \right\|_p^p + \left\| \sum_{i \ne j} d_i F(i,j) - F(j,j) \right\|_p^p \right) \\
&\ge \sum_{d \in M_j} 2\|F(j,j)\|_p^p = 2^n \|F(j,j)\|_p^p\,;
\end{aligned}$$

here, we are using the fact that the function $t \mapsto t^p$ is increasing and convex on $\mathbb{R}^+$. This holds for each $j \in \mathbb{N}_n$, and so, summing over j, we see that

$$2^n \sum_{j=1}^{n} \|F(j,j)\|_p^p \le \sum_{j=1}^{n} \sum_{d \in D_n} \|x_{j,d}\|_p^p \le 2^n C^p\,.$$

Hence we have $\sum_{j=1}^{n} \|F(j,j)\|_p^p \le C^p$, and the result follows. ☐

For a measurable subset V of Ω and $U \in \mathcal{B}(L^1(\Omega), L^p(\Omega))$, we define $\chi_V U \in \mathcal{B}(L^1(\Omega), L^p(\Omega))$ by the formula

$$(\chi_V U)(a)(s) = \chi_V(s) U(a)(s) \quad (a \in L^1(G),\ s \in G)\,.$$

Proposition 9.4. *Let $\{X_1, \dots, X_n\}$ and $\{Y_1, \dots, Y_n\}$ be measurable partitions of Ω. Then, for each bounded linear operator $R : \mathcal{B}(L^1(\Omega), L^p(\Omega)) \to L^p(\Omega)$, we have*

$$\left(\sum_{i=1}^{n} \|\chi_{X_i} R(\chi_{Y_i} U)\|_p^p \right)^{1/p} \le \|R\| \|U\| \qquad (U \in \mathcal{B}(L^1(\Omega), L^p(\Omega)))\,.$$

Proof. Take $U \in \mathcal{B}(L^1(\Omega), L^p(\Omega))$, and define $F : \mathbb{N}_n \times \mathbb{N}_n \to L^p(\Omega)$ by

$$F(i,j) = \chi_{X_j} R(\chi_{Y_i} U) \quad (i,j \in \mathbb{N}_n)\,.$$

For each $(d_1, \dots, d_n) \in D_n$, we have

$$\begin{aligned}\sum_{j=1}^n \left\| \sum_{i=1}^n d_i F(i,j) \right\|_p^p &= \sum_{j=1}^n \left\| \sum_{i=1}^n d_i \chi_{X_j} R(\chi_{Y_i} U) \right\|_p^p = \sum_{j=1}^n \left\| \chi_{X_j} R\left(\sum_{i=1}^n d_i \chi_{Y_i} U \right) \right\|_p^p \\ &= \left\| R\left(\sum_{i=1}^n d_i \chi_{Y_i} U \right) \right\|_p^p \leq \|R\|^p \left\| \sum_{i=1}^n d_i \chi_{Y_i} U \right\|^p = \|R\|^p \|U\|^p\,,\end{aligned}$$

and so, by Lemma 9.3, we have

$$\left(\sum_{j=1}^n \|F(j,j)\|_p^p \right)^{1/p} = \left(\sum_{j=1}^n \|\chi_{X_j} R(\chi_{Y_j} U)\|_p^p \right)^{1/p} \leq \|R\| \|U\|\,,$$

which gives the result. ☐

In the following result, we are regarding $\sum_{i=1}^n U'(f_i) \otimes x_i$ as a finite–rank operator from $L^1(\Omega)$ to $L^p(\Omega)$ by using equation (1). We recall that q is the conjugate index to p.

Lemma 9.5. *Let $U \in \mathcal{B}(L^1(\Omega), L^p(\Omega))$, let $f_1, \dots, f_n \in L^q(\Omega)$ have pairwise–disjoint supports, and let $x_1, \dots, x_n \in L^p(\Omega)$ have pairwise–disjoint supports. Set*

$$T = \sum_{i=1}^n U'(f_i) \otimes x_i : L^1(\Omega) \to L^p(\Omega)\,.$$

Then $T \in \mathcal{B}(L^1(\Omega), L^p(\Omega))$ and $\|T\| \leq \|U\| \max\{\|f_i\|_q \|x_i\|_p : i \in \mathbb{N}_n\}$.

Proof. Set $X_i = \operatorname{supp} f_i \ (i \in \mathbb{N}_n)$ and $C = \max\{\|f_i\|_q \|x_i\|_p : i \in \mathbb{N}_n\}$. For each $a \in L^1(\Omega)$, we have

$$\begin{aligned}\|Ta\|_p^p &= \left\| \sum_{i=1}^n \langle Ua, f_i \rangle\, x_i \right\|_p^p = \sum_{i=1}^n |\langle Ua, f_i \rangle|^p \, \|x_i\|_p^p \\ &\leq \sum_{i=1}^n \|\chi_{X_i} U(a)\|_p^p \|f_i\|_q^p \|x_i\|_p^p \leq C^p \|\chi_{X_1 \cup \cdots \cup X_n} U(a)\|_p^p \leq C^p \|Ua\|_p^p\,.\end{aligned}$$

Therefore $\|Ta\|_p \leq C\|Ua\|_p$, and the result follows. ☐

In the theorem below, we shall use the following identity. For each $x \in L^p(G)$, $\lambda \in L^\infty(G)$, and $s \in G$, we have

$$(\lambda \cdot s) \otimes x = s^{-1} * [\lambda \otimes (s \cdot x)]\,. \tag{12}$$

Theorem 9.6. *Let G be a locally compact group, and take $p \in (1, \infty)$. Then $L^p(G)$ is injective in $L^1(G)$-**mod** if and only if G is amenable.*

Proof. It is well-known that, if G is amenable, then $L^p(G)$ is injective: by Johnson's theorem [14], $L^1(G)$ is an amenable Banach algebra, and explicitly by [11, VII.2.29] E' is injective for each $E \in$ **mod**-$L^1(G)$. Hence $L^p(G)$ is injective. Thus we need to consider only the converse. So we suppose that $L^p(G)$ is injective in $L^1(G)$-**mod**; we may also suppose that G is infinite.

Recall that we are setting $J = \mathcal{B}(L^1(G), L^p(G))$ and $q = p'$. By Proposition 9.2, there is a morphism $R \in {}_{L^1(G)}\mathcal{B}(\widetilde{J}, L^p(G))$ with $R \circ \widetilde{\Pi} = I_{L^p(G)}$. For each compact subset V of G with $m(V) > 0$, we define a linear functional Λ_V on $L^\infty(G)$ by

$$\langle \lambda, \Lambda_V \rangle = \frac{1}{m(V)} \int_V (R(\lambda \otimes \chi_V))(t)\,\mathrm{d}m(t) \quad (\lambda \in L^\infty(G))\,.$$

For each $\lambda \in L^\infty(G)$, we have

$$|\langle \lambda, \Lambda_V \rangle| \le \|R(\lambda \otimes \chi_V)\|_p \|\chi_V / m(V)\|_q \le \|R\| \|\lambda\|_\infty \|\chi_V\|_p \|\chi_V / m(V)\|_q = \|R\| \|\lambda\|_\infty\,,$$

and so $\Lambda_V \in L^\infty(G)'$ with $\|\Lambda_V\| \le \|R\|$. Let $\mathcal{V}$ be the family of compact neighbourhoods of the identity e in G, and set $V_1 \le V_2$ if $V_2 \subset V_1$. Then $(\mathcal{V}, \le)$ is a directed set. Let Λ be a weak* accumulation point in $L^\infty(G)'$ of the bounded net $\{\Lambda_V : V \in \mathcal{V}\}$. By passing to a subnet, we may suppose that $\{\Lambda_V : V \in \mathcal{V}\}$ converges to Λ. Clearly $\langle 1, \Lambda \rangle = 1$ since, for each $V \in \mathcal{V}$, we have

$$\langle 1, \Lambda_V \rangle = \frac{1}{m(V)} \int_V (R(\widetilde{\Pi}\chi_V))(t)\,\mathrm{d}m(t) = \frac{1}{m(V)} \int_V \mathrm{d}m(t) = 1\,,$$

and so Λ is non-zero. We *claim* that Λ is left (p,p)-multi-invariant.

Take $n \in \mathbb{N}$, and consider distinct elements $s_1, \dots, s_n$ of G. Choose $V \in \mathcal{V}$ such that the sets $s_1 V, \dots, s_n V$ are pairwise disjoint. Let $U \in J$, and let $\mathbf{X} = \{X_1, \dots, X_n\}$ be a measurable partition of G. Take $f_1, \dots, f_n \in L^q(G)_{[1]}$ with $\operatorname{supp} f_i \subset X_i$ $(i \in \mathbb{N}_n)$, and finally set

$$T = \sum_{i=1}^n U'(f_i) \otimes \chi_{s_i V} : L^1(G) \to L^p(G)\,.$$

By Lemma 9.5, $T \in J$ and $\|T\| \le \|U\| m(V)^{1/p}$.

For each $i \in \mathbb{N}_n$, we have

$$\begin{aligned} m(V)\left\langle f_i, U''(s_i \cdot \Lambda_V)\right\rangle &= m(V)\left\langle U'(f_i), s_i \cdot \Lambda_V\right\rangle \\ &= \int_V R((U'(f_i)\cdot s_i) \otimes \chi_V)(t)\,\mathrm{d}m(t) \\ &= \int_V R(U'(f_i) \otimes (s_i \cdot \chi_V))(s_i t)\,\mathrm{d}m(t) \\ &= \int_{s_i V} R(U'(f_i) \otimes \chi_{s_i V})(t)\,\mathrm{d}m(t) \\ &= \int_{s_i V} R(\chi_{s_i V} T)(t)\,\mathrm{d}m(t)\,; \end{aligned}$$

the third equality holds true by (12) and because R is a $L^1(G)$-**mod** homomorphism from $\widetilde{J}$ into $L^p(G)$ and $L^p(G)$ is essential in $L^1(G)$-**mod**. Hence, by Hölder's inequality, we have

$$\left|\left\langle U'(f_i), s_i \cdot \Lambda_V\right\rangle\right| \le \|\chi_{s_i V} R(\chi_{s_i V} T)\|_p m(V)^{\frac{1}{q}-1}\,.$$

Then, by Proposition 9.4, we have

$$\begin{aligned} \left(\sum_{i=1}^n \left|\left\langle f_i, U''(s_i \cdot \Lambda_V)\right\rangle\right|^p\right)^{1/p} &\le \left(\sum_{i=1}^n \|\chi_{s_i V} R(\chi_{s_i V} T)\|_p^p\right)^{1/p} m(V)^{\frac{1}{q}-1} \\ &\le \|R\| \|T\| m(V)^{\frac{1}{q}-1} \le \|R\| \|U\| m(V)^{\frac{1}{p}} m(V)^{\frac{1}{q}-1} = \|R\| \|U\|\,. \end{aligned}$$

Therefore

$$\left(\sum_{i=1}^{n}\left|\left\langle f_i, U''(s_i\cdot\Lambda)\right\rangle\right|^p\right)^{1/p} = \lim_V\left(\sum_{i=1}^{n}\left|\left\langle U'(f_i), s_i\cdot\Lambda_V\right\rangle\right|^p\right)^{1/p} \leq \|R\|\,.$$

Since this is true for all such families $\{f_1,\dots,f_n\}$ in $L^q(G)_{[1]}$, we have

$$\left(\sum_{i=1}^{n}\|\chi_{X_i}U''(s_i\cdot\Lambda)\|_p^p\right)^{1/p} \leq \|R\|\,.$$

Since this is true for each measurable partition $\mathbf{X}=\{X_1,\dots,X_n\}$ and each $U\in J_{[1]}$, it follows from Proposition 7.4 that

$$\|(s_1\cdot\Lambda,\dots,s_n\cdot\Lambda)\|_n^{(p,p)} \leq \|R\|\,.$$

Thus Λ is a non-zero, left (p,p)-multi-invariant functional on $L^\infty(G)$, and so, by Lemma 8.2 and Theorem 8.4, the group G is amenable. □

The determination of when $L^p(G)$ is flat in the category $L^1(G)$-**mod** is an easy consequence of the previous theorem and the following simple observation.

Let A be a Banach algebra, and suppose that $\theta : A\to A$ is a Banach algebra anti-automorphism. For each Banach left A-module E, we can define a Banach right A-module E_θ as follows. As a Banach space $E_\theta = E$, and the right A-module action on E_θ is defined as

$$x\cdot a := \theta(a)\cdot x \qquad (a\in A, x\in E_\theta)\,.$$

In fact, it is obvious that every Banach right A-module has the form E_θ for some suitable $E\in A$-**mod**. By going through the definition directly, we obtain the following.

Lemma 9.7. *Let E be a Banach left A-module. Then E is injective in A-**mod** if and only if E_θ is injective in **mod**-A.* □

Consider again the locally compact group G, and take $p\in(1,\infty)$, with conjugate index q. We see that the dual right $L^1(G)$-module action $\cdot$ on $L^q(G) = (L^p(G),\star)'$ is given by

$$(h\cdot f)(t) = \int h(st)f(s)\,\mathrm{d}m(s) = (\widetilde{f}\star h)(t) \qquad (f\in L^1(G),\ h\in L^q(G))\,,$$

where $\widetilde{f}(t) = f(t^{-1})\Delta(t^{-1})$ for $f\in L^1(G)$, so that $\widetilde{f}\in L^1(G)$. Thus $(L^p(G),\star)' = (L^q(G),\star)_\theta$ where $\theta : f\mapsto \widetilde{f}$ is an isometric anti-automorphism on $L^1(G)$.

Theorem 9.8. *Let G be a locally compact group, and take $p\in(1,\infty)$. Then $L^p(G)$ is flat in $L^1(G)$-**mod** if and only if G is amenable.*

Proof. We know that $(L^p(G),\star)$ is flat in $L^1(G)$-**mod** if and only if the dual module $(L^p(G),\star)' = (L^q(G),\star)_\theta$ is injective in **mod**-$L^1(G)$, and hence if and only if $(L^q(G),\star)$ is injective in $L^1(G)$-**mod**. By the main theorem, this holds if and only if G is amenable. □

In summary, by combining Theorems 8.4, 9.6, and 9.8, we obtain the following theorem.

Theorem 9.9. *Let G be a locally compact group, and take $p\in(1,\infty)$. Then the following conditions are equivalent:*

1. *G is amenable;*
2. *$L^p(G)$ is injective in $L^1(G)$-**mod**;*

3. *$L^p(G)$ is flat in $L^1(G)$-**mod**;*

4. *G is left (p,q)-amenable for all $q \geq p$;*

5. *G is left (p,q)-amenable for some $q \geq p$;*

6. *G is left $(1,q)$-amenable for all $q \geq 1$;*

7. *G is left $(1,q)$-amenable for some $q \geq 1$.* □

Remark 9.10. We also have that, for each $p \in (1,\infty)$, $L^p(G)$ is [injective / flat] in the categories [**mod**-$L^1(G)$ / $L^1(G)$-**mod**-$L^1(G)$] if and only if G is amenable.

Remark 9.11. There are natural quantitative versions of projectivity, injectivity, and flatness. These were first explicitly introduced and studied in [24]. Let A be a Banach algebra, and let $E \in A$-**mod** be injective. We set

$$\mathrm{inj}(E) = \inf \|\rho\|\,,$$

where the infimum is taken over all right-inverse morphisms ρ to the canonical morphism Π.

It follows from the previous theorem that, for a locally compact group G and $p \in (1,\infty)$, we have $\mathrm{inj}(L^p(G)) = 1$ whenever $L^p(G)$ is injective in $L^1(G)$-**mod**.

Recently, G. Racher [19] has proved (by different methods to us) that a discrete group G is amenable whenever $\ell^{\,2}(G)$ is injective in $\ell^{\,1}(G)$-**mod** with $\mathrm{inj}(\ell^{\,2}(G)) = 1$.

10 Semigroup algebras

Let S be a semigroup, with product denoted by juxtaposition. We recall that S is: (i) *left-cancellative* if the map $L_s : t \mapsto st,\ S \to S$, is injective for each $s \in S$; (ii) *weakly left-cancellative* if $\{u \in S : su = t\}$ is finite for each $s, t \in S$; (ii) *uniformly weakly left-cancellative* if

$$\sup_{s,t \in S} |\{u \in S : su = t\}| < \infty\,.$$

Further, S is *right-cancellative* if the map $R_s : t \mapsto ts,\ S \to S$, is injective for each $s \in S$, and S is *cancellative* if it is both left- and right-cancellative.

Let S be a semigroup. Then the Banach space $(\ell^{\,1}(S), \|\cdot\|_1)$ is a Banach algebra with respect to a product $\star$ satisfying the condition that $\delta_s \star \delta_t = \delta_{st}$ for each $s,t \in S$. The Banach algebra $(\ell^{\,1}(S), \|\cdot\|_1, \star)$ is the *semigroup* algebra of S; for a discussion of this algebra, see [4].

Let S be a semigroup. The action of S on $\ell^{\,1}(S)$ is defined by $s \cdot f := \delta_s \star f$, so that

$$(s \cdot f)(t) = \sum_{sr=t} f(r) \quad (t \in S)$$

for $s \in S$ and $f \in \ell^{\,1}(S)$. This action can be extended by duality first to an action of S on the space $\ell^{\,1}(S)' = \ell^{\,\infty}(S)$ and then to an action on $\ell^{\,\infty}(S)' = \ell^{\,1}(S)''$. This latter extended action is denoted by

$$(s, \Lambda) \mapsto s \cdot \Lambda, \quad S \times \ell^{\,\infty}(S)' \to \ell^{\,\infty}(S)'\,.$$

In the case where S is left-cancellative, the action of each $s \in S$ on $\ell^{\,1}(S)$ is an isometry, and so its extension to an action on $\ell^{\,\infty}(S)'$ is also an isometry.

The following proposition is easily checked.

Proposition 10.1. *Let S be a semigroup, and take $p > 1$. Then $\ell^{\,p}(S)$ is a Banach left $\ell^{\,1}(S)$-module if and only if S is uniformly weakly left-cancellative.* □

Suppose now that S is a non-empty set, and take $p \geq 1$. Then we set $J = \mathcal{B}(\ell^1(S), \ell^p(S))$. It is easy to see that J can be identified isometrically with the Banach space

$$\ell^{\infty,p}(S) := \left\{ U : S \times S \to \mathbb{C} \colon \|U\| = \sup_{s \in S} \left(\sum_{t \in S} |U(s,t)|^p \right)^{1/p} < \infty \right\};$$

the identification is given by $U(s,t) = U(\delta_s)(t) \quad (s,t \in G)$.

Lemma 10.2. *Let S be a non-empty set, and take $p \geq 1$. Suppose that $R : \ell^{\infty,p}(S) \to \ell^p(S)$ is a bounded linear operator. Further, suppose that U and U_s $(s \in S)$ are in $\ell^{\infty,p}(S)$ and are such that $|U_s(r,t)| \leq |\delta_s(t)U(r,t)|$ $(s,r,t \in S)$. Then*

$$\left(\sum_{s \in S} |R(U_s)(s)|^p \right)^{1/p} \leq \|U\| \|R\| .$$

Proof. This follows from (the proof of) Proposition 9.4. □

Now suppose that S is a uniformly weakly left-cancellative semigroup. Define a left $\ell^1(S)$-module action on $\ell^{\infty,p}(S)$ by

$$(f * U)(s,t) := \sum \{ f(r)U(x,y) : \ r,x,y \in S, \ rx = s, ry = t \} \qquad (f \in \ell^1(S), \ U \in \ell^{\infty,p}(S)) .$$

This induces a new Banach left $\ell^1(S)$-module action on J similar to the one given in (11); to avoid confusion with the standard module action on J, we shall denote this new module by $(\widetilde{J}, *)$. We shall freely identify $\ell^{\infty,p}(S)$ with $(\widetilde{J}, *)$. Define $\widetilde{\Pi} : \ell^p(S) \to \widetilde{J}$ by

$$\widetilde{\Pi}(g)(f) = \sum_{s \in S} f(s)g \qquad (f \in \ell^1(S), \ g \in \ell^p(S)) ;$$

in the identification $\widetilde{J} = \ell^{\infty,p}(S)$, we have $\widetilde{\Pi}(g)(s,t) = g(t)$. It is then obvious that $\widetilde{\Pi}$ is an admissible left $\ell^1(S)$-module homomorphism.

Again, the next proposition follows from the definition of injectivity.

Proposition 10.3. *Let S be a uniformly weakly left-cancellative semigroup, and take $p \geq 1$. Suppose that $\ell^p(S)$ is injective in $\ell^1(S)$-**mod**. Then the morphism $\widetilde{\Pi} : \ell^p(S) \to \widetilde{J}$ is a coretraction.* □

We shall also need the following, taken from [21, Proposition 3.1].

Proposition 10.4. *Let A be a Banach algebra, and let E be injective in A-**mod**. Then, for every $Q \in \mathcal{B}(A, E)$ and $D \subset A$ such that $Q(ab) = aQ(b)$ for all $a \in A$ and $b \in D$, there exists $x_0 \in E$ with $Q(b) = bx_0$ for all $b \in D$.* □

Let S be a semigroup. Then an element $\Lambda \in \ell^\infty(S)'$ is a *mean* on $\ell^\infty(S)$ if $\langle 1, \Lambda \rangle = \|\Lambda\| = 1$, and Λ is *left-invariant* if $\{s \cdot \Lambda : \ s \in S\} = \{\Lambda\}$; the semigroup S is *left-amenable* if there exists a left-invariant mean on $\ell^\infty(S)$. See [17, p. 16] for more details.

Definition 10.5. A functional $\Lambda \in \ell^\infty(S)'$ is *left (p,q)−multi-invariant* if the set $\{s \cdot \Lambda : \ s \in S\}$ is multi-bounded in the (p,q)-multi-norm.

Theorem 10.6. *Let S be a left-cancellative semigroup, and take $p \geq 1$. Suppose that $\ell^p(S)$ is injective in $\ell^1(S)$-**mod**. Then S is left-amenable and has a right identity.*

Proof. Set $A = \ell^1(S)$ and $E = \ell^p(S)$, and consider the natural injection $\iota : A \to E$. Then ι is obviously a left A-module homomorphism. By Proposition 10.4, there exists $g_0 \in E$ such that $f = f \star g_0$ for every $f \in A$. Since S is left-cancellative, it follows easily that S has a right identity, say e.

By Proposition 10.3, there exists $R \in {}_A\mathcal{B}(\widetilde{J}, E)$ with $R \circ \widetilde{\Pi} = I_E$. We define $\Lambda \in \ell^\infty(S)'$ by

$$\langle \lambda, \Lambda \rangle = (R(\lambda \otimes \delta_e))(e) \quad (\lambda \in \ell^\infty(S)),$$

where $\lambda \otimes \delta_e \in \mathcal{B}(\ell^1(S), \ell^p(S)) = \widetilde{J}$, as in (1). We have

$$\langle 1, \Lambda \rangle = (R(\chi_S \otimes \delta_e))(e) = (R(\widetilde{\Pi}\delta_e))(e) = \delta_e(e) = 1.$$

We *claim* that Λ is left (p,p)-multi-invariant.

Take $n \in \mathbb{N}$, and consider distinct elements $s_1, \ldots, s_n$ of S. Let $U \in \mathcal{B}(A, E)$, and take $f_1, \ldots, f_n \in E'_{[1]}$ with pairwise–disjoint supports. Define

$$T = \sum_{i=1}^{n} U'(f_i) \otimes \delta_{s_i} : A \to E.$$

By Lemma 9.5, $T \in J$ and $\|T\| \le \|U\|$.

For each $i \in \mathbb{N}_n$, since S is left-cancellative, we have

$$\begin{aligned}\left\langle f_i, U''(s_i \cdot \Lambda)\right\rangle &= \left\langle U'(f_i), s_i \cdot \Lambda\right\rangle = R((U'(f_i) \cdot s_i) \otimes \delta_e)(e)\\ &= R[\delta_{s_i} * ((U'(f_i) \cdot s_i) \otimes \delta_e)](s_i) = R(T_i)(s_i),\end{aligned}$$

where we set $T_i = \delta_{s_i} * ((U'(f_i) \cdot s_i) \otimes \delta_e)$. We see that, for each $r, t \in S$, we have

$$\begin{aligned}T_i(r,t) &= \sum \left\{((U'(f_i) \cdot s_i) \otimes \delta_e)(x,y) : \ x, y \in S, \ s_i x = r, \ s_i y = t\right\}\\ &= \sum \left\{U'(f_i)(s_i x)\delta_e(y) : \ x, y \in S, \ s_i x = r, \ s_i y = t\right\}\\ &= \sum \left\{U'(f_i)(r)\delta_{s_i}(t) : \ x, y \in S, \ s_i x = r, \ s_i y = t\right\}\\ &= \sum \left\{\delta_{s_i}(t)T(r,t) : \ x, y \in S, \ s_i x = r, \ s_i y = t\right\},\end{aligned}$$

and so $T_i(r,t)$ is either $\delta_{s_i}(t)T(r,t)$ or 0. Thus $|T_i| \le |\delta_{s_i}T|$; here, we are identifying J with $\ell^{\infty,p}(S)$. Hence, by Lemma 10.2, we have

$$\left(\sum_{i=1}^{n} \left|\left\langle f_i, U''(s_i \cdot \Lambda)\right\rangle\right|^p\right)^{1/p} = \left(\sum_{i=1}^{n} |R(T_i)(s_i)|^p\right)^{1/p} \le \|R\|\|U\|.$$

Since this is true for all such collections $\{f_1, \ldots, f_n\}$ in $E'_{[1]}$, by Proposition 7.4 when $p > 1$ and the same calculation as in the proof of Proposition 7.4 when $p = 1$, we have

$$\|(s_1 \cdot \Lambda, \ldots, s_n \cdot \Lambda)\|_n^{(p,p)} \le \|R\|.$$

Therefore the set $\{s \cdot \Lambda \colon s \in S\}$ is (p,p)-multi-bounded.

Note that, since S is left-cancellative, the map $L_s : \Psi \mapsto s \cdot \Psi$ on $\ell^\infty(S)'$ is isometric (as well as being positive), and so $|s \cdot \Lambda| = s \cdot |\Lambda|$ for each $s \in S$. Thus, essentially as in Lemma 8.2, we see that $|\Lambda| / \|\Lambda\|$ is a left (p,p)-multi-invariant mean on $\ell^\infty(S)$. We can then argue in a similar way to that in the proof of Theorem 8.4 by applying the Ryll-Nardzewski fixed point theorem to the semigroup $\{L_s : \ s \in S\}$ to find a left-invariant mean on $\ell^\infty(S)$. $\square$

The following theorem in the case where $p = 1$ was proved in [21, Theorem 4.10].

Theorem 10.7. *Let S be a cancellative semigroup, and take $p \in [1, \infty)$. Then $\ell^p(S)$ is injective in $\ell^1(S)$-**mod** if and only if S is an amenable group.*

Proof. Certainly, $\ell^p(S)$ is injective in $\ell^1(S)$-**mod** whenever S is an amenable group.

Suppose that $\ell^p(S)$ is injective in $\ell^1(S)$-**mod**. By Theorem 10.6, S is left-amenable and has a right identity. It remains to prove that S is a group; the argument is similar to that in [21, Theorem 4.10]. Since S is cancellative, a right identity e of S must be the identity of S. For each $t \in S$, we consider a map $Q_t : \ell^1(S) \to \ell^p(S)$ defined as

$$Q_t \colon \sum_{s \in S} \alpha_s \delta_s \mapsto \sum_{s \in S} \alpha_{st} \delta_s \,,$$

so that $Q_t(\delta_{st}) = \delta_s \ (s \in S)$. Since R_t is injective on S, we have $Q_t(f \star \delta_{st}) = f \star Q_t(\delta_{st})$ for all $f \in \ell^1(S)$ and $s \in S$. By Proposition 10.4, there exists $a_t \in \ell^p(S)$ such that

$$Q_t(\delta_{st}) = \delta_{st} \star a_t \quad (s \in S)\,.$$

Thus $\delta_s = \delta_{st} \star a_t \ (s \in S)$; in particular, $\delta_e = \delta_t \star a_t$. This implies that there exists $u \in S$ with $tu = e$. Now $utu = ue = eu$, and so the injectivity of R_u implies that $ut = e$. Hence u is the inverse of t in S. This shows that S is an (amenable) group. $\square$

Corollary 10.8. *Let S be a right–cancellative semigroup, and take $p \in [1, \infty)$. Suppose that $\ell^p(S)$ is flat in $\ell^1(S)$-**mod**. Then S is right–amenable and has a left identity. If, furthermore, S is cancellative, then S is an amenable group.* $\square$

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